Large Deviations Principles for McKean Vlasov Equations under Hölder Norms

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McKean Vlasov SDE's

A **McKean Vlasov** Stochastic Differential Equation is one where the coefficients are dependent on the law of the process.

Consider the following example:

$$X(t) = x + \int_0^t \mathbb{E}[X(s)]ds + W(t)$$

Here $\mathbb{E}[X(t)] = xe^t$ so the solution of the SDE is $X(t) = xe^t + W(t)$.

The intervention of the law in the trajectory of the process makes these equations nonlinear. These equations are used to model continuous **mean-field systems** of large numbers of particles with interactions.

Existence of McKean Vlasov solutions relies on coefficients of SDE being Lipschitz with respect to the **Wasserstein Metric**

$$W^{(2)}(\mu,\nu) = \inf_{\pi\in\mathcal{P}_2(E\times E)} \left(\int_{E\times E} |x-y|^2 d\pi(x,y) \right)^{\frac{1}{2}}$$

where $\pi(x,y)$ has marginals $\mu(x)$ and $\nu(y)$. See [Carmona, 2016].

The proof involves constructing a **Contraction Operator** on the space of measures on continuous paths and using Banach Fixed Point.

Large Deviation Principles

We are interested in the behaviour of **Stochastic Differential Equations** where the driving noise is controlled by some asymptotic parameter ε which is going to zero.

For instance in a dynamic system which is subject to very small random perturbations, it would be valuable to understand whether the **deterministic system** is a **good approximation** of a more complicated stochastic system.

We use tools from **Freidlin-Wentzell Theory** which treat the solutions of SDEs as a path valued random variable to prove LDP results for different metrics on path space.

The **Cameron Martin Space** H is the set of all absolutely continuous paths $h(t) = \int_0^t \dot{h}(s)ds$ where $\dot{h} \in L^2([0,1])$.

We call the solution to the ODE which approximates the SDE the **Skeleton Process**. The Skeleton replaces the driving Wiener process by a smooth deterministic element of the Cameron Martin Space.

Large Deviation Principles are well understood for a supremum path norm. We wish to extend these results to other pathspace norms with a **coarser topology**, in particular the Hölder norm.

Hölder Norms

For $\alpha < 0.5$, we define the Hölder norms to be

$$||f||_\alpha = ||f||_\infty + \sup_{t,s\in[0,1]} \frac{|f(t)-f(s)|}{|t-s|^\alpha}$$

We can prove upper bounds of the tails of the distributions of the α-norm of the following stochastic processes:

$$\mathbb{P}\Big\{ \| \int_0^\cdot K(s)dW(s)\|_\alpha \geq u, \|K\|_\infty \leq 1 \Big\} \leq C' \exp\left(\frac{-u^2}{C'}\right) \tag{1}$$

$$\mathbb{P}\Big\{ \|W\|_\alpha \geq u, \|W\|_\infty \leq v \Big\} \leq C \max\left(1, \left(\frac{u}{v}\right)^{1/\alpha}\right) \exp\left(\frac{-1}{C}\frac{u^{1/\alpha}}{v^{(1/\alpha)-2}}\right) \tag{2}$$

The Main Results

Let $h \in H$. We prove Large Deviation Principles in Hölder Norms for the class of SDEs

$$X_\varepsilon^x(t) = x + \int_0^t b_\varepsilon(s, X_\varepsilon^x(s), \mathcal{L}(X_\varepsilon^x(s)))ds + \varepsilon \int_0^t \sigma_\varepsilon(s, X_\varepsilon^x(s), \mathcal{L}(X_\varepsilon^x(s)))dW(s)$$

with Skeleton Process

$$\Phi^x(h)(t) = x + \int_0^t b(s, \Phi^x(h)(s), \delta_{\Phi^x(h)(s)})ds + \int_0^t \sigma(s, \Phi^x(h)(s), \delta_{\Phi^x(h)(s)})\dot{h}(s)ds$$

where $(b_\varepsilon, \sigma_\varepsilon)(t,x,\mu)$ converges uniformly to $(b,\sigma)(t,x,\mu)$. b has monotone growth in x and Lipschitz in μ. σ is bounded and Lipschitz in both x and μ. b and σ are both continuous in time. Note $\varepsilon W(t)$ is replaced by $h(t)$.

LDP for Hölder Norms

Using methods from [Arous, 1994], it is well understood one can extend LDP results from a supremum norms to other pathspace norms by proving the following:

Theorem

$\forall R, \rho > 0\ \exists \delta, \nu > 0$ such that $\forall 0 < \varepsilon < \nu$,

$$\mathbb{P}\Big\{ ||X_\varepsilon^x - \Phi^x(h)||_\alpha \geq \rho, ||\varepsilon W - h||_\infty \leq \delta \Big\} \lesssim \exp\left(\frac{-R}{\varepsilon^2}\right)$$

This represents an upper bound on the probability that the Stochastic Process deviates from the Skeleton Process and the random perturbation does not deviate much from its drift term.

Proof

Firstly, we prove the inequality when the drift coefficients $b = 0$ and $h = 0$. Let $X_\varepsilon^{x,l}$ be a step function approximation of the process X_ε^x on a net of size $1/l$. Then

$$\begin{aligned}
&\mathbb{P}\Big\{ \|\varepsilon \int_0^\cdot \sigma(X_\varepsilon^x(s), \mathcal{L}(X_\varepsilon^x(s)))dW(s)\|_\alpha \geq \rho, \|\varepsilon W\|_\infty \leq \delta \Big\} \\
&\leq \mathbb{P}\Big\{ \|\varepsilon \int_0^\cdot [\sigma(X_\varepsilon^x(s), \mathcal{L}(X_\varepsilon^x(s))) - \sigma(X_\varepsilon^{x,l}, \mathcal{L}(X_\varepsilon^{x,l}))]dW(s)\|_\alpha \geq \frac{\rho}{2}, \\
&\qquad \|X_\varepsilon^x - X_\varepsilon^{x,l}\|_\infty + \mathbb{E}[\|X_\varepsilon^x - X_\varepsilon^{x,l}\|_\infty^2]^{1/2} \leq \gamma \Big\} && (3) \\
&\quad + \mathbb{P}\Big\{ \|X_\varepsilon^x - X_\varepsilon^{x,l}\|_\infty + \mathbb{E}[\|X_\varepsilon^x - X_\varepsilon^{x,l}\|_\infty^2]^{1/2} > \gamma, \|X_\varepsilon^x\|_\infty < N \Big\} && (4) \\
&\quad + \mathbb{P}\Big\{ \|\varepsilon \int_0^\cdot \sigma(X_\varepsilon^{x,l}(s), \mathcal{L}(X_\varepsilon^{x,l}(s)))dW(s)\|_\alpha \geq \frac{\rho}{2}, \|\varepsilon W\|_\infty \leq \delta \Big\} && (5) \\
&\quad + \mathbb{P}\Big\{ \|X_\varepsilon^x\|_\infty \geq N \Big\} && (6) \\
&\lesssim \exp\left(\frac{-R}{\varepsilon^2}\right) && \square
\end{aligned}$$

Equation (3) is controlled by Equation (1) and Lipschitz of σ. Equation (4) is controlled by uniform continuity of the process X_ε^x. Equation (5) is controlled by Equation (2) using a discretization argument. Equation (6) is controlled using LDP for the Process under a supremum norm.

Extending the argument to general drift b involves using **Grönwall's Inequality** and a **Girsanov** measure change.

Functional Iterated Logarithm Law for McKean Vlasov Equations

Let $\Gamma_\alpha : \mathbb{R}^d \to \mathbb{R}^d$, a collection of continuous bijections, be a System of Contractions centered at x. Consider the SDE

$$dY(t) = \sigma\Big(Y(t), \mathcal{L}(Y(t))\Big)dW(t) + b(Y(t), \mathcal{L}(Y(t)))dt \qquad Y(0) = x$$

and $Z_u(t) = \Gamma_{\phi(u)}(Y(ut))$. Then Z_u satisfies the SDE

$$dZ_u(t) = \frac{1}{\sqrt{\log\log(u)}}\hat{\sigma}_u\Big(Z_u(t), \mathcal{L}(Z_u(t))\Big)d\mathcal{W}_u(t) + \hat{b}_u\Big(Z_u(t), \mathcal{L}(Z_u(t))\Big)dt$$

with coefficients

$$\hat{\sigma}_u(x,\mu) = \phi(u)\nabla\Big[\Gamma_{\phi(u)}\Big]\Big(\Gamma_{\phi(u)^{-1}}(x)\Big)^T \sigma\Big(\Gamma_{\phi(u)^{-1}}(x), \mu\circ\Gamma_{\phi(u)}\Big)$$

$$\hat{b}_u(x,\mu) = u\mathbf{L}(x,\mu)\Big[\Gamma_{\phi(u)}\Big]\Big(\Gamma_{\phi(u)^{-1}}(x)\Big),$$

$\mathbf{L}$ is the generator of Y and $\mathcal{W}_u$ is a rescaled Brownian motion. Let us assume that $\hat{\sigma}_u$ and $\hat{b}_u$ converge uniformly to some $\hat{\sigma}$ and $\hat{b}$.

Then we apply our Large Deviation Principles Result to get

Theorem

With Probability 1, the set of paths $\{Z_u; u > 3\}$ is relatively compact in the α-Hölder topology and its set of limit points coincides with $K = \{\Phi(h) : \frac{\|\dot{h}\|_2^2}{2} \leq 1\}$.

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