Non-linear Reflected Stochastic Differential Equation

Romain Ravaille, on a joint work [4]

Introduction

We consider the mean-field particles system defined by:

$$dX_t^i = \sigma dB_t - \nabla V(X_t^i)dt - \frac{1}{N}\sum_{j=1}^{N} \nabla F(X_t^i - X_t^j)dt - dk_t^i$$

$$|k^i|_t = \int_0^t \mathbb{1}_{\partial\mathcal{D}}(X_s^i)d|k|_s \qquad k_t^i = \int_0^t (X_s^i)d|k^i|_s, \qquad \text{with } \mathcal{D} \in \mathbb{R}^d. \tag{1}$$

With some classical hypotheses, we get the coupling (propagation of chaos):

$$\sup_{t\in[0;T]} \mathbb{E}\left[\left|\overline{X_t^1} - X_t^1\right|^2\right] \leq \frac{K_2(T)}{N},$$

with $\overline{X_t^1}$ being the solution of the Non-Linear Reflected Stochastic Differental Equation:

$$X_t = X_0 + \sigma B_t - \int_0^t \nabla V(X_s)ds - \int_0^t \nabla F * u(s, Xs)ds - k_t \tag{2}$$

with $\mathbb{P}[X_t \in dx] = u(t, dx)$, $|k|_t = \int_0^t \mathbb{1}_{\partial\mathcal{D}}(X_s)d|k|_s$, $k_t = \int_0^t \mathbf{n}_s d|k|_s$.

and $\nabla F * u$ being the convolution product between ∇F and u.

Existence and uniqueness of a solution

Steps of the proof (inspired by [1]) that Equation (2) admits a unique strong solution on $\mathbb{R}_+$:

1. We start by TANAKA's [3] Proof of the existence of a strong solution for the Linear Reflected Stochastic Differential Equation.
2. We define an operator with fixed points that are solutions of Equation (2).
3. We prove that, on a well chosen interval $[0, T]$, there exists a unique fixed point for the operator by proving it is a contraction.
4. We then engage in *reductio ad absurdum* to prove that the result extends to $\mathbb{R}_+$.

The reflection lets us bypass the moments' control we would usually need.

Description of the invariant probabilities

First, by $\mathcal{S}_\sigma$, we denote the set of invariant probabilities for Equation (2).
Some computation on Equation (2) gives us the following Fokker-Planck equation:

$$\frac{\partial}{\partial t}u(t,x) = \nabla\left[\frac{\sigma^2}{2}\nabla u(t,x) + u(t,x)\Big(\nabla V(x) + \nabla F * u(t,x)\Big)\right],$$

$$\frac{\sigma^2}{2}\nabla u(t,y) = -[(\nabla F * u(t,\cdot) + \nabla V(.))u(t,\cdot)](y),$$

for $y \in \partial\mathcal{D}$ e.g. a von Neumann boundary condition.
We can easily prove that:

1. For any diffusion coefficient $\sigma > 0$, $\mathcal{S}_\sigma \neq \emptyset$.
2. If we assume moreover that the external potential is symmetric, we obtain the existence of a symmetric invariant probability measure.

Case with V convex

1. We can have the uniform propagation of chaos (same as above but not depending on T).
2. We take Z^i solution of Equation (1) with the same Brownian motion as X^i, μ_t to be the law of $\overline{X_t^1}$ starting at μ_0 and ν_t the law of $\overline{Z_t^1}$ starting at ν_0. We consider $\mathbb{W}_2(\mu_t, \nu_t)$ the Wasserstein distance between μ_t and ν_t. Using the uniform propagation of chaos, we have:

$$\mathbb{W}_2(\mu_t, \nu_t) \leq \mathbb{W}_2(\mu_t, \nu_t^{1,N}) + \mathbb{W}_2(\mu_t^{1,N}, \nu_t^{1,N}) + \mathbb{W}_2(\mu_t^{1,N}, \nu_t)$$
$$\leq \frac{2\sqrt{K}}{\sqrt{N}} + \mathbb{W}_2(\mu_t^1, \nu_t^{1,N}),$$

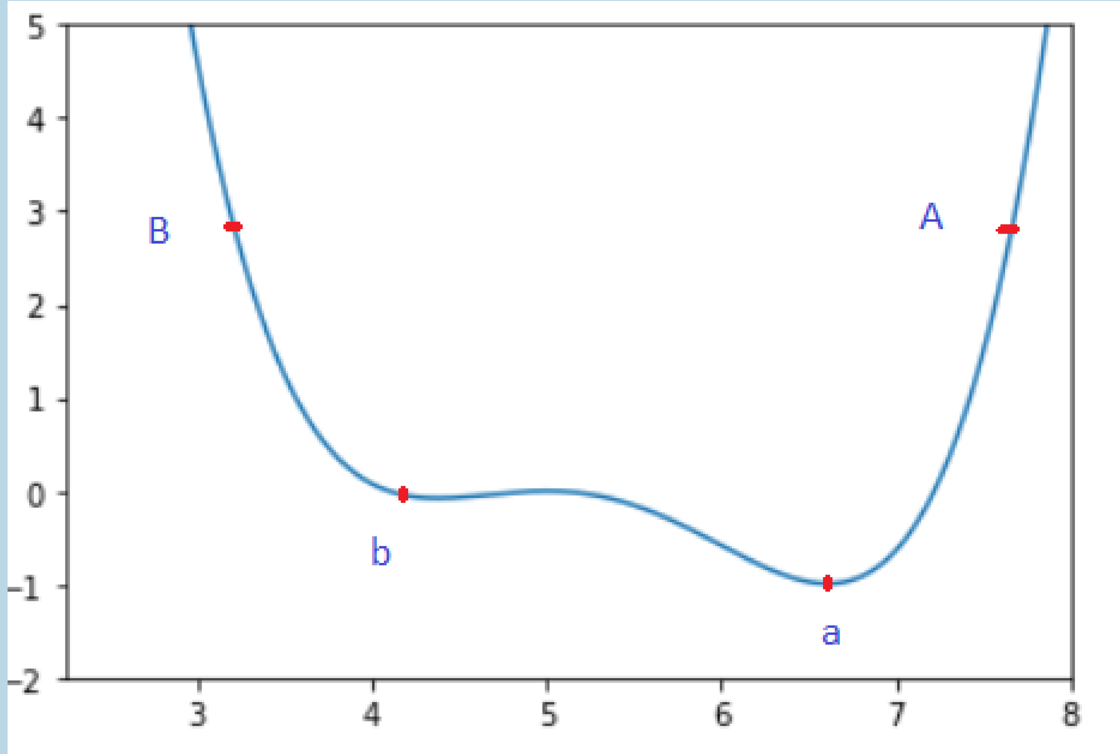
with $\nu_t^{1,N}$ the law of Z_t^1 starting at ν_0, $\nu_t^{1,N}$ the law of Z_t^1 starting at ν_0. From that, by studying $\mathbb{W}_2(\mu_t^{1,N}, \nu_t^{1,N})$, we have the exponential convergence rate when N tends to infinity: $\mathbb{W}_2(\mu_t, \nu_t) \leq \mathbb{W}_2(\mu_0, \nu_0)e^{-2\lambda t}$.

Case with V non-convex

1. We no longer have the uniqueness of invariant probabilities and uniform propagation of chaos.
2. As in [6] we have that the number of elements in $\mathcal{S}_\sigma$ depends on σ.

An application

Equation (2) can also be seen as an arguably better version of the gradient descent algorithm. It would provide the right solution if one tries to find the minimum of the following function with B as a starting point:



In this case, the reflection is providing an interval of interest that lets the algorithm find the right answer faster.

Handling the reflection term

The following reasoning is used multiple times during this work to dominate certain reflection terms:
If $x \in \partial\mathcal{D}$ and $\mathbf{n}_x$ the exiting normal vector in x, then for each $y \in \mathcal{D}$ we have

$$\langle y - x, \mathbf{n}_x\rangle \leq 0.$$

*Let $x \in \partial\mathcal{D}$. With [5] we know that there exists a halfplane that contains the set $\mathcal{D}$. We obtain a hyperplane $\mathcal{H}_x$ that contains the point x but does not contain any of the interior points of $\mathcal{D}$.
Then, $\forall y \in \mathcal{D} \subset \mathcal{H}_x$ we have that $y - x$ is pointing toward $\mathcal{H}_x$ while $\mathbf{n}_x$ is by definition pointing toward the outside of $\mathcal{H}_x$. This leads to our inequality.*

References

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[4] dos Reis, G. , Ravaille, R. , Salkeld, W. and Tugaut, J., *Non-linear reflected self-stabilizing diffusion in convex and non-convex setting*, work in progress, 2019.

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