A model for Self-Propelled diffusions

M. Ottobre (Maxwell Institute and Heriot Watt University, Edinburgh)

Joint work with P. Butta (La Sapienza), F. Flandoli (Scuola Normale Superiore),
B. Zegarlinski (Imperial College)

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Space-homogeneous case $f_t = f_t(v)$

$$\partial_t f_t = -G(M_1(t))\partial_v f_t + \partial_v(v f_t) + \sigma\,\partial_{vv} f_t(v)$$

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- You can close an incredibly simple equation on the cumulants

$$\dot{C}_n(t) = -nC_n(t) \qquad \text{for } n \geq 3\,.$$

- Entropy functional

$$S(f) := \int_{\mathbb{R}} \mathrm{d}v \left[f(v) \log f(v) + \frac{v^2}{2\sigma} f(v) \right] + \frac{1}{\sigma} V(M_1(t))$$

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- Exponential decay comes with a rate

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Then

$$f_0 = \mu_+ \qquad \text{and} \quad f_n = 0 \;\; \forall n \geq 1\,.$$