Asymptotic behaviour of some self-interacting diffusions

Aline Kurtzmann
joint works with V. Kleptsyn, P. Del Moral and J. Tugaut

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Outline

1. Generalities

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2. Self-attracting diffusion on $\mathbb{R}$ (with Victor Kleptsyn)

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What is a self-interacting diffusion?

- Solution of

$$\mathrm{d}X_t = \mathrm{d}B_t - F(t, X_t, \mu_t)\mathrm{d}t$$

- $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s$

Brownian polymer

Durrett and Rogers (1992) on $\mathbb{R}^d$:

$$\mathrm{d}X_t = \mathrm{d}B_t + \int_0^t f(X_t - X_s)\,\mathrm{d}s\,\mathrm{d}t,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is mesurable and bounded.

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Question: what is the normalisation of X?

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- Herrmann & Roynette (2003):

Theorem (Herrmann & Roynette, 2003)

1) Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function, decreasing and bounded. Suppose that there exists $C, \rho > 0$ and $k \in \mathbb{N}^*$ such that $|f(x)| \geq Ce^{-\rho/|x|^k}$ around 0. Then X_t converges a.s.
2) When the interaction is not local, $f(x) = -sign(x)\mathbf{1}_{\{|x|\geq a\}}$, then the trajectories remain bounded a.s. (but do not converge).

Reinforced diffusion on a compact set

Benaïm, Ledoux and Raimond (2002), Benaïm and Raimond (2003, 2005) on a compact manifold:

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{t}\int_0^t \nabla_x W(X_t, X_s)\mathrm{d}s\,\mathrm{d}t$$

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Heuristic: show that $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s$ is close to a deterministic flow.

Outline

Study

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}B_t - \left(\frac{1}{t} \int_0^t W'(X_t - X_s)\,\mathrm{d}s \right) \mathrm{d}t \\ &= \mathrm{d}B_t - W' * \mu_t(X_t)\mathrm{d}t \end{aligned}$$

$$\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} \mathrm{d}s$$

Example: quadratic W

Lemma

Let $W(x) = ax^2$ with $a > 0$. Then a.s. the empirical measure μ_t converges (weakly) to $\mu_\infty \sim \mathcal{N}(\bar{\mu}_\infty, 1/a)$.

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Let $W(x) = \frac{1}{2}(x-1)^2$. Then $\overline{\mu}_t = \frac{1}{t}\int_0^t X_s \mathrm{d}s$ and X_t diverge a.s.

Set of hypotheses on the interaction potential (H)

- W is $\mathcal{C}^2$, strictly uniformly convex and symmetric,

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- W is $\mathcal{C}^2$, strictly uniformly convex and symmetric,
- there exist $C, k > 0$ such that

$$|W(x)| + |W'(x)| + |W''(x)| \leq C(1 + |x|^k).$$

Results

Theorem

*Suppose that W satisfies the assumption **(H)**. Then there exists a unique probability density function ρ_∞ such that a.s.*

$$\mu_t \to \rho_\infty(\cdot - c_\infty)\mathrm{d}x.$$

Outline

Relation with a Markovian system

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μ_t is asymptotically close to the deterministic dynamical system:
$\dot{\mu} = \Pi(\mu) - \mu$,

where $\Pi(\mu) := \frac{1}{Z(\mu)} e^{-2W*\mu}$.

Strategy of the proof

- Compare on $[T_n, T_{n+1}]$ the trajectories of

$$\mathrm{d}X_t = \mathrm{d}B_t - W' * \mu_t(X_t)\mathrm{d}t$$

with those of the corresponding process where μ_t is replaced by μ_{T_n}:

$$\mathrm{d}Y_t = \mathrm{d}B_t - W' * \mu_{T_n}(Y_t)\mathrm{d}t$$

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- Estimate the speed of convergence of the empirical measure of Y toward the invariant probability measure $\Pi(\mu_{T_n})$

Strategy for the approximation by a dynamical system

- Compare the flow obtained by the “Euler method”

$$\mu_{[T_n,T_{n+1}]} = \mu_{T_n} + \frac{\Delta T_n}{T_{n+1}} \left(\mu_{[T_n,T_{n+1}]} - \mu_{T_n} + \text{error}\right)$$

with the flow

$$\dot{\mu} = \frac{1}{T_n}(\Pi(\mu) - \mu)$$

Outline

Reference point

Definition

The center of the probability measure is the point c_μ such that $W' * \mu(c_\mu) = 0$.
We define the centered measure μ^c as

$$\mu^c(A) = \mu(A + c_\mu).$$

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- Lyapunov function: free energy
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- Convergence of the center

Final result

Theorem (Kleptsyn-K., EJP 2012)

*Suppose that W satisfies the hypothese **(H)**. Then:*

1. *there exists a unique probability density function ρ_∞ centered such that a.s.*

$$\mu_t^c \to \rho_\infty(x)\mathrm{d}x,$$

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2. *a.s. the center $c_t = c(\mu_t)$ converges to a (random) limit c_∞.*

And there exists $a > 0$ such that a.s., we have for t large enough

$$\mathbb{W}_2(\mu_t^c, \rho_\infty) = O(\exp\{-a \sqrt[2k+1]{\log t}\}).$$

Outline

The considered self-interacting diffusion

$$\begin{aligned} \mathrm{d}X_t &= \sigma \mathrm{d}B_t - \left(V'(X_t) + \frac{1}{t}\int_0^t W'(X_t - X_s)\,\mathrm{d}s \right) \mathrm{d}t \\ &= \sigma \mathrm{d}B_t - \left(V'(X_t) + W' * \mu_t(X_t) \right) \mathrm{d}t \end{aligned}$$

$$\begin{aligned} \mu_t &= \frac{1}{t}\int_0^t \delta_{X_s} \mathrm{d}s \\ \sigma &> 0. \end{aligned}$$

Final result

Theorem (Del Moral- K. -Tugaut, submitted 2019)

*Suppose that V and W satisfy the hypothese **(H)**. Let m be the unique minimum of V.*
Let $\mathcal{D}$ be a stable domain for the flow $x \mapsto -V'(x) - W'(x-m)$.
Denote by τ the first time the process X exits the domain $\mathcal{D}$.
Let $H = \inf_{x \in \partial\mathcal{D}} V(x) + W(x-m) - V(m)$ be the exit cost from $\mathcal{D}$.

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Then we have for any $\delta > 0$

$$\lim_{\sigma\to 0} \mathbb{P}(e^{2(H-\delta)/\sigma^2} \leq \tau \leq e^{2(H+\delta)/\sigma^2}) = 1.$$

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- Probability of leaving a stable domain before the empirical measure remains stuck in $\mathbb{B}(\delta_m, \kappa)$ goes to zero with σ
- Coupling between X and a Markov diffusion to use former Tugaut's results