Propagation of chaos results for some McKean-Vlasov models

Jean-François Jabir

School of Mathematics, University of Edinburgh (← 01/07/2019),
National research University HSE, Moscow (→ 30/06/2019).

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Models of interest:

- Prototypical McKean-Vlasov dynamic:

$$dX_t = \Big(\int b(X_t - x)\mu(t, dx) \Big)\, dt + \sigma dW_t, \;\; \mu(t) = \mathcal{L}(X_t);$$

- Langevin McKean-Vlasov dynamic:

$$\begin{cases} dY_t = V_t\, dt, \\ dV_t = \Big(\int b(Y_t - y, V_t - v)\, \mu(t, dy, dv) \Big)\, dt + \sigma dW_t. \end{cases}$$

- Conditional McKean Lagrangian model:

$$\begin{cases} dY_t = V_t\, dt, \\ dV_t = \mathbb{E}\left[b(V_t) \,|\, Y_t \right]\, dt + \sigma dW_t. \end{cases}$$

Generically, σ is a positive scalar or $\sigma \in \mathbb{R}^{d \times d}$ positive definite, and b measurable bounded.

Toy models (Sznitman 1991): Given $0 < T < \infty$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ bounded Lipschitz continuous, $(X_0^i, (W_t^i;\, t \geq 0))$ independent copies of $(X_0, (W_t;\, t \geq 0))$,

$$X_t^{i,N} = X_0^i + \int_0^t \frac{1}{N} \sum_{j=1}^{N} b(X_s^{i,N} - X_s^{j,N})\, ds + \sigma\, W_t^i,$$

$$X_t^{i,\infty} = X_0^i + \int_0^t \left(\int b(X_s^{i,\infty} - y)\, \mu(s, dy) \right) ds + \sigma\, W_t^i,\ \mu(t) = \mathcal{L}(X_t^{i,\infty}).$$

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$$X_t^{i,\infty} = X_0^i + \int_0^t \left(\int b(X_s^{i,\infty} - y)\, \mu(s, dy) \right) ds + \sigma\, W_t^i,\ \mu(t) = \mathcal{L}(X_t^{i,\infty}).$$

Theorem 1 (Sznitman 1989)

There exists $0 < C < \infty$, which doesn't depend on N, such that, for all $1 \leq i \leq N$,

$$W_1\left(\mathcal{L}(X_t^{i,N};\, t \leq T), \mathcal{L}(X_t^{i,\infty};\, t \leq T)\right) \leq \mathbb{E}\left[\max_{0 \leq t \leq T} \left| X_t^{i,N} - X_t^{i,\infty} \right|\right] \leq \frac{C}{\sqrt{N}},$$

where W_1 is the (truncated) Monge-Kantorovich-Wasserstein distance:

$$W_1(P, Q) = \inf_{\pi \text{ coupling of } P \text{ and } Q} \int_{\mathcal{C}([0,T];\mathbb{R}^d) \times \mathcal{C}([0,T];\mathbb{R}^d)} (|x - y| \wedge 1)\, \pi(dx, dy).$$

Examples in the literature:

- Probabilistic interpretation of nonlinear pdes/stochastic particle methods :
  - Propagation of chaos for stochastic particle system with smooth or moderate interaction kernel: Funaki 1984; Oelschläger 1984, 1985; Léonard 1986; ... Méléard and Roelly-Coppoletta 1987; Méléard 1995; Bossy and Talay 1996; Jourdain and Méléard 1998; Antonelli and Kohastu-Higa 2002;
  - Burgers equation: Sznitman 1986; Bossy and Talay 1997; Jourdain 1997; ...
  - Incompressible Navier-Stokes equations: Marchioro and Pulvirenti 1982; Osada 1984; Méléard 2001; Fontbona 2004; ...
  - Conservative equations: Bossy and Jourdain 2000; Jourdain 2002; Jourdain and Reygner 2013; ...
  - Viscous Pressureless gas equation: Dermoune 2001; ...
  - Chemotaxis Keller-Segel model: Fournier and Jourdain 2017; Cattiaux and Pédéches 2017; Tomašević 2018; ...
- Mean Field Games and Optimal control problems for McKean-Vlasov dynamics (differential calculus on the space of probability measures): Lasry and Lions 2006; Lions "Cours au Collège de France" (2006-2012); Cardaliaguet *et al.* 2013; ...; Carmona and Delarue 2013, 2015, 2018; ...
- Recent quantitative "weak" propagation of chaos results: For $\psi : \mathbb{R}^d \to \mathbb{R}$ some test function,

$$\left|\mathbb{E}[\psi(X_T^{i,N})] - \mathbb{E}[\psi(X_T)]\right| \leq \frac{C}{N}.$$

  Ref.: Kolokoltsov 2010; Kolokoltsov, Troeva and Wang 2014 ; Mischler and Mouhot 2013; Mischler, Mouhot and Wennberg 2014; Jourdain and Bencheikh 2019; Chassagneux, Szpruch and Tse 2019; Chaudru de Raynal and Frikha 2019, ...

Propagation of chaos result for general Vlasov models with non-smooth interaction kernel: Hauray and Jabin 2015, Jabin and Wang 2016, 2017.

- **Interacting particle systems with bounded interacting kernel** b:

$$\begin{cases} Y_t^{i,N} = Y_0^{i,N} + \int_0^t V_s^{i,N}\, ds, \\ V_t^{i,N} = V_0^{i,N} + \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \int_0^t b(Y_s^{i,N} - Y_s^{j,N})\, ds + \sigma W_t^i. \end{cases}$$

- **Mean field limit system**:

$$\begin{cases} Y_t = Y_0 + \int_0^t V_s\, ds, \\ Y_t = V_0 + \int_0^t \left(\int b(Y_s - x) \mu(s, dy) \right) ds + \sigma W_t, \\ \mathcal{L}(Y_t, V_t) = \mu(t). \end{cases}$$

Main approach: PDE analysis.

Propagation of chaos result for general Vlasov models with non-smooth interaction kernel: Hauray and Jabin 2015, Jabin and Wang 2016, 2017.

- **Interacting particle systems with bounded interacting kernel** b: For $\mathbf{x}^N = (x_1, \ldots, x_N) \in \mathbb{R}^d \times \ldots \mathbb{R}^d$, $\mathbf{u}^N = (u_1, \ldots, u_N) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d$,

$$\begin{cases} \partial_t f^N + \mathbf{u}^N \cdot \nabla_{\mathbf{x}^N} f^N + \dfrac{1}{N-1} \displaystyle\sum_{j=1, j\neq i}^{N} b(x_i - x_j) \cdot \nabla_{u_i} f^N + \dfrac{\sigma^2}{2} \triangle_{\mathbf{u}^N} f^N = 0 \\ f^N(0, \mathbf{x}^N, \mathbf{u}^N) = f_0(\mathbf{x}^N, \mathbf{u}^N). \end{cases}$$

- **Related McKean-Vlasov model**: Related Fokker-Planck equation:

$$\begin{cases} \partial_t f + u \cdot \nabla_x f + \displaystyle\int b(x-y) f(t, y, v)\, dy\, dv \cdot \nabla_u f + \dfrac{\sigma^2}{2} \triangle_u f = 0, \\ f(0, x, u) = f_0(x, u). \end{cases}$$

- Propagation of chaos result in terms of the total variation distance:

$$\|\mu - \nu\|_{TV, \mathcal{P}(\mathbb{R}^d)} = \left| \sup_{A \in \mathcal{B}(\mathbb{R}^d)} (\mu(A) - \nu(A)) \right|, \ \mu, \nu \in \mathcal{P}(\mathbb{R}^d),$$

and the relative entropy (provided the density $d\mu/d\nu$ of μ w.r.t. ν exists):

$$H(\mu \,|\, \nu) = \int \log(d\mu/d\nu)(z)\mu(dz).$$

Propagation of chaos result: Define $f^{k,N}$ as the marginal p.d.f. of $(Y_t^{i,N}, V_t^{i,N})$, $i = 1, \dots, k$:

$$f^{k,N}(t, x_1, \dots, x_k, u_1, \dots, u_k) = \int f^N(t, x_1, \dots, x_k, y_{k+1}, \dots, y_N, u_1, \dots, u_k, v_{k+1}, \dots, v_N)$$

and $f^{k,\infty}$ as the marginal p.d.f. of (Y_t, V_t), $i = 1, \dots, k$:

$$f^{k,\infty}(t, x_1, \dots, x_k, u_1, \dots, u_k) = f(t, x_1, u_1) \times \cdots \times f(t, x_k, u_k).$$

Theorem 2 (Jabin and Wang 2016)

Assume independency of the initial positions, and that for some $\lambda > 0$,

$$\int_{(\mathbb{R}^d \times \mathbb{R}^d)} f_0(x, u) \left(|x|^{2\lambda} + |u|^{2\lambda} + \ln(f_0(x, u)) \right) < \infty.$$

Then

$$\forall t \geq 0, \ \|f^{k,N}(t) - f^{k,\infty}(t)\|_{L^1} \leq \sqrt{2H(f^{k,N}(t)|f^{k,\infty}(t))} \leq C/\sqrt{N}.$$

Element of proof:

◦ Preliminary steps: By Csiszár-Pinsker-Kullback inequality,

$$\|f^{k,N}(t) - f^{k,\infty}(t)\|_{L^1} \leq \sqrt{2H(f^{k,N}(t)|f^{k,\infty}(t))}.$$

Next, by the super-additive property of the renormalized relative entropy (see Hauray and Mischler 2014),

$$\frac{1}{k}H(f^{k,N}(t)|f^{k,\infty}(t)) \leq \frac{1}{N}H(f^{N,N}(t)|f^{N,\infty}(t)).$$

◦ Entropy estimate: Use the Fokker-Planck equation related to f^N to get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{N}H(f^{N,N}(t)|f^{N,\infty}(t))\right)\\ &\leq \frac{1}{N}H(f^{N,N}(t)|f^{N,\infty}(t)) + \frac{1}{N}\int f^{N,\infty}(t,\mathbf{x}^N,\mathbf{u}^N)\exp\{|R_N(t,\mathbf{x}^N,\mathbf{u}^N)|\}\,d\mathbf{x}^N\,d\mathbf{u}^N\end{aligned}$$

where

$$R_N(t,\mathbf{y}^N,\mathbf{v}^N) = \frac{1}{N}\sum_{i,j=1}^{N}\left(\nabla_{v_i}\log(f^{N,\infty})\cdot\left(b(y_i-y_j) - \int b(y_i-y)f(t,y,v)\,dv\right)\right)$$

The rest of the proof relies on the estimate
$\sup_N \int f^{N,\infty}(t,\mathbf{x}^N,\mathbf{u}^N)\exp\{|R_N(t,\mathbf{x}^N,\mathbf{u}^N)|\}\,d\mathbf{x}^N\,d\mathbf{u}^N < \infty$ and Gronwall's lemma.

• Jabin and Wang 2018: For $b \in W^{-1,\infty}$ (i.e. $b^{(k)}(x) = \sum_l \partial_{x_l} G^{k,l}(x)$, $G \in L^\infty$) on the d-dimensional torus $\mathbb{T}^d$:

$$X_t = \left(Z_0 + \int_0^t \left\{ F(Z_s) + \int b(Z_s - y)\mu(s, dy) \right\} ds + \sigma W_t \right) \text{mod}1, \; X_t \sim \mu(t),$$

$$X_t^{i,N} = \left(X_0^i + \frac{1}{N} \sum_{j=1, j\neq i}^{N} \int_0^t \left\{ F(X_s^{i,N}) + b(X_s^{i,N} - X^{j,N}) \right\} ds + \sigma W_t \right) \text{mod}1,$$

Theorem 3 (Jabin and Wang 2018)

Assuming that $F, \nabla_x \cdot F$ are in L^∞, b, $\nabla_x \cdot b \in W^{-1,\infty}$, that the Fokker-Planck equation related to $(X_t;\, t \leq T)$ admits a positive density in $L^\infty((0,T); W^{2,p}(\mathbb{T}^d))$ for all $p < \infty$ and that the initial entropy $H(f^N(0)|f^{N,\infty}(0))$ satisfies the following properties:

$$\sup_{N \to \infty} NH(f^N(0)|f^{N,\infty}(0)) < \infty.$$

Then

$$ess\,sup_{t \leq T} \|f^{k,N}(t) - f^{k,\infty}(t)\|_{L^1} \leq \frac{C}{\sqrt{N}}.$$

Starting point for a probabilistic analog: Bounded interaction and non-degenerated case: $\sigma \in \mathbb{R}^{d\times d}$ constant invertible:

$$X_t = X_0 + \int_0^t \int b(X_s - y)\mu(s, dy)\, ds + \sigma W_t,\; X_t \sim \mu(t). \tag{1}$$

Corollary 4 (A corollary of Mishura and Veretennikov 2016)

Assume that $\mathbb{E}[|Z_0|^4] < \infty$, $b : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel function and $\sigma \in \mathbb{R}^{d\times d}$ is invertible. Then (1) admits at most one unique weak solution.

Elements of proof: Let

$$\widehat{X}_t = \widehat{X}_0 + \int_0^t \int b(\widehat{X}_s - y)\mu(s, dy)\, ds + \sigma \widehat{W}_t,\; \widehat{X}_t \sim \mu(t).$$

be another solution to (1) and let P_T and $\widehat{P}_T$ be the laws of $(X_t)_{0\leq t\leq T}$ and $(\widehat{X}_t)_{0\leq t\leq T}$. Then, again, by the Csiszar-Pinsker-Kullback inequality,

$$\|P_T - \widehat{P}_T\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))} \leq \sqrt{2H(P_T \mid \widehat{P}_T)} \left(= \sqrt{2\mathbb{E}_{P_T}[\log(dP_T/d\widehat{P}_T)]}\right).$$

As

$$dP_T/d\widehat{P}_T = \exp\Big\{ -\int_0^T \sigma^{-1}\Big(b(X_t-y)\mu(t,dy) - \int b(X_t-y)\widehat{\mu}(t,dy)\Big)\, dW_t$$
$$+\frac{1}{2}\int_0^T \Big|\sigma^{-1}\Big(b(X_t-y)\mu(t,dy) - \int b(X_t-y)\widehat{\mu}(t,dy)\Big)\Big|^2\, dt\Big\},$$

where $(W_t;\ t\leq T)$ is a Brownian motion under P_T, it follows that

$$H(P_T\,|\,\widehat{P}_T) = \frac{1}{2}\mathbb{E}[\int_0^T \Big|\sigma^{-1}\Big(\int b(X_t-y)\mu(t,dy) - \int b(X_t-y)\widehat{\mu}(t,dy)\Big)\Big|^2\, dt].$$

Since the above is bounded by $\|b\|_{L^\infty}\int_0^T \max_{s\leq t}\|\mu(s)-\widehat{\mu}(s)\|^2\, dt$, uniqueness follows by Gronwall's inequality.

Toward a propagation of chaos result: Instead of considering two solutions of a McKean-Vlasov dynamic, what would happen if we consider N independent copies of

$$dX_t^{i,\infty} = \Big(\int b(X_t^{i,\infty}-y)\mu(t,\,dy)\Big)\, dt + \sigma\, dW_t^i,$$

and the related interacting particle system:

$$dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^N b(X_t^{i,N}-X_t^{j,N})\, dt + \sigma\, dW_t^i\ ?$$

Answer for path-dependent McKean-Vlasov dynamics: Lacker 2018 (also Veretennikov April 2018 - Arxiv - for similar ideas on the queueing system GI/GI/1):

$$dX_t^{i,\infty} = B\big(t, (X_r^{i,\infty})_{0\le r\le t}, \mathcal{L}((X_r^{i,\infty})_{0\le r\le t})\big)\, dt + A\big(t, (X_r^{i,\infty})_{0\le r\le t}\big)\, dW_t^i,$$

$$dX_t^{i,N} = B\Big(t, (X_r^{i,N})_{0\le r\le t}, \frac{1}{N}\sum_{i=1}^{N} \delta_{\{(X_r^{i,N})_{0\le r\le t}\}}\Big)\, dt + A\big(t, (X_r^{i,N})_{0\le r\le t}\big)\, dW_t^i,$$

Theorem 5 (Propagation of chaos, Lacker 2018)

Assume that A is invertible and that the SDE $d\chi_t = A(t, (\chi_r)_{0\le r\le t})\, dW_t$, $\chi_0 \sim \mu_0$ has a unique strong solution. Assume also that $A^{-1}B$ is bounded,

$$\big|(A^{-1}B)(t,z,\mu) - (A^{-1}B)(t,z,\nu)\big| \le K\|\mu-\nu\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))}$$

and: for all $\mu \in \mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))$,

$$\nu \mapsto \int_{\mathcal{C}([0,T];\mathbb{R}^d)} \int_0^T \big|A^{-1}(t,z)\,(B(t,z,\mu) - B(t,z,\nu))\big|^2\, dt\, \nu(dz)$$

is sequentially continuous. Then the following propagation of chaos holds:

$$\lim_{N\to\infty} \|P^{k,N} - P^{k,\infty}\|_{TV,\mathcal{P}((\mathcal{C}([0,T];\mathbb{R}^{kd})))} = \lim_{N\to\infty} H(P^{k,\infty} \,|\, P^{k,N}) = 0,$$

for

$$P^{k,\infty} = \mathbb{P}\circ((X_t^{1,\infty},\dots,X_t^{k,\infty})_{t\le T})^{-1},\ P^{k,N} = \mathbb{P}\circ((X_t^{1,N},\dots,X_t^{k,N})_{t\le T})^{-1}.$$

Elements of proof for the TV and entropy limits: Preliminary, by a LDP, show that for all set $U \in \mathcal{B}(\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d)))$ containing $\mathcal{L}((X_t)_{0\leq t\leq T})$,

$$\lim_{N\to\infty} \mathbb{P}\Big(\Gamma_t^N \notin U\Big) = 0,\ \Gamma_t^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\{(X_r^{i,N})_{0\leq r\leq t}\}}.$$

Next, by Csiszár-Pinsker-Kullback inequality,

$$\|P^{k,N} - P^{k,\infty}\|_{TV,\mathcal{P}((\mathcal{C}([0,T];\mathbb{R}^{kd})))} \leq \sqrt{2H(P^{k,\infty}\mid P^{k,N})}$$

where $H(P^{k,\infty}\mid P^{k,N})$ is the relative entropy linked to introduce the interacting kernel in the k first components of the McKean-Vlasov model. Using the exchangeability of the partially-interacting particle system, Girsanov's transformation and appropriate L^p-estimates on the density $dP^{k,\infty}/dP^{k,N}$,

$$H(P^{k,\infty}\mid P^{k,N}) \leq \frac{k\|A^{-1}B\|_{L^\infty}\, Te^{4kT\|A^{-1}B\|_{L^\infty}}}{2}\times$$
$$\left(\mathbb{E}[\int_0^T \left|(A^{-1}B)\left(t,(X_s^{1,N})_{0\leq s\leq t},\Gamma_t^N\right) - (A^{-1}B)(t,(X_r^{1,N})_{0\leq r\leq t},\mathcal{L}((X_r^{1,\infty})_{0\leq r\leq t}))\right|^2 dt]\right)^{1/2}$$

The regularity assumptions on A^{-1} and B ensures that the upper-bound tends to 0 as $N\to\infty$.

Toward a quantitative propagation of chaos result: Using the estimate

$$\|P^{k,N} - P^{k,\infty}\|_{TV,\mathcal{P}((\mathcal{C}([0,T];\mathbb{R}^{kd})))} \leq \sqrt{2H(P^{k,N}\,|\,P^{k,\infty})}$$

and the supper-additivity property of the relative entropy on $\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))$:

$$H(P^{k,N}\,|\,P^{k,\infty}) = \frac{k}{N}H(P^{N,N}\,|\,P^{N,\infty}).$$

$\Rightarrow$ Provided that $\max_{N\geq 1} H(P^{N,N}\,|\,P^{N,\infty}) < \infty$, an (optimal) quantitative propagation of chaos emerges.
Sufficient condition: For $\delta > 0$,

$$\mathbb{E}\left[(Z_T^{N,\infty})^{1+\delta}\right] < \infty,$$

where $Z_T^{N,\infty}$ is the exponential martingale allowing to pass from the N-McKean-Vlasov model to the N-interacting particle system:

$$Z_T^{N,\infty} := dP^{N,N}/dP^{N,\infty} = \exp\left\{-\sum_{i=1}^{N}\int_0^T \Delta B_t^{i,N}\cdot dW_t^i - \frac{1}{2}\int_0^T\sum_{i=1}^{N}\left|\Delta B_t^{i,N}\right|^2 dt\right\},$$

$$\Delta B_t^{i,N} = A^{-1}(t,(X_r^{i,\infty})_{0\leq r\leq t}) \times\left(B\big(t,(X_r^{i,\infty})_{0\leq r\leq t},\frac{1}{N}\sum_{j=1}^{N}\delta_{\{(X_r^{j,\infty})_{0\leq r\leq t}\}}\big) - B\big(t,(X_r^{i,\infty})_{0\leq r\leq t},\mathcal{L}((X_r^{i,\infty})_{0\leq r\leq t})\big)\right).$$

Alternative: Coming back to the definition of the total variation distance between $P^{k,\infty}$ and $P^{k,N}$:

$$\begin{aligned}
&\|P^{k,N}-P^{k,\infty}\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))}\\
&=\sup_{A\in\mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^{kd}))}\left|\mathbb{P}((X^{1,N},\dots,X^{k,N})\in A)-\mathbb{P}((X^{1,\infty},\dots,X^{k,\infty})\in A)\right|\\
&=\sup_{A\in\mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^{kd}))}\left|\mathbb{E}[Z_T^{N,\infty}\,\mathbb{1}_{\{(X^{1,\infty},\dots,X^{k,\infty})\in A\}}]-\mathbb{P}((X^{1,\infty},\dots,X^{k,\infty})\in A)\right|\\
&=\sup_{A\in\mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^{kd}))}\left|\mathbb{E}\left[\left(Z_T^{N,\infty}-1\right)\mathbb{1}_{\{(X^{1,\infty},\dots,X^{k,\infty})\in A\}}\right]\right|\\
&=\sup_{A\in\mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^{kd}))}\left|\mathbb{E}\left[\sum_{i=1}^{N}\int_0^T Z_t^{N,\infty}\Delta B_t^{i,N}\,dW_t^i\mathbb{1}_{\{(X^{1,\infty},\dots,X^{k,\infty})\in A\}}\right]\right|\\
&\qquad\text{(applying Itô's formula)}\\
&=\sup_{A\in\mathcal{B}(\mathcal{C}([0,T];\mathbb{R}^{kd}))}\left|\mathbb{E}\left[\sum_{i=1}^{k}\int_0^T Z_t^{N,\infty}\Delta B_t^{i,N}\,dW_t^i\mathbb{1}_{\{(X^{1,\infty},\dots,X^{k,\infty})\in A\}}\right]\right|\\
&\qquad\text{(since }(X_t^{1,\infty},W_t^1;\,t\le T),\dots,(X_t^{N,\infty},W_t^N;\,t\le T)\text{ are i.i.d.)}\\
&=\mathbb{E}\left[\left|\mathbb{E}\left[\sum_{i=1}^{k}\int_0^T Z_t^{N,\infty}\Delta B_t^{i,N}\,dW_t^i\,\middle|\,(X^{1,\infty},\dots,X^{k,\infty})\right]\right|\right].
\end{aligned}$$

An illustrative example: Taking the prototypical case: Diffusion A constant, b bounded,

$$X_t^{i,N} = X_0^i + \int_0^t \frac{1}{N}\sum_{j=1}^N b(X_s^{i,N} - X_s^{j,N})\,ds + \sigma\,W_t^i,$$

$$X_t^{i,\infty} = X_0^i + \int_0^t \left(\int b(X_s^{i,\infty} - y)\,\mu(s,dy)\right)\,ds + \sigma\,W_t^i,\ \mu(t) = \mathcal{L}(X_t^{i,\infty}).$$

Then,

$$\Delta B_t^{i,N} = \sigma^{-1}\left(\frac{1}{N}\sum_{j=1}^N b(X_t^{i,\infty} - X_t^{j,\infty}) - \int b(X_t^{i,\infty} - y)\mu(t,dy)\right)$$

Rough estimate: For $k=1$, using BDG and Hölder inequalities,

$$\begin{aligned}\|P^{k,N} - P^{k,\infty}\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))} &= \mathbb{E}\left[\left|\mathbb{E}\left[\int_0^T Z_t^{N,\infty}\Delta B_t^{i,N}\,dW_t^i \,\middle|\, (X^{1,\infty},\ldots,X^{k,\infty})\right]\right|\right] \\ &\leq \sum_{i=1}^{k}\left(\mathbb{E}\left[\int_0^T (Z_t^{N,\infty})^4\,dt\right]\right)^{1/4}\left(\mathbb{E}\left[\int_0^T |\Delta B_t^{i,N}|^4\,dt\right]\right)^{1/4}.\end{aligned}$$

Since $(X_t^{1,\infty};\ t\leq T),\ldots,(X_t^{N,\infty};\ t\leq T)$ are i.i.d. $\mathbb{E}\left[|\Delta B_t^{i,N}|^4\right] \leq \frac{C_T}{N^2}$ and, provided that $\mathbb{E}[(Z_T^{N,\infty})^4] < \infty$ one can deduce an explicit rate of convergence:

$$\|P^{k,N} - P^{k,\infty}\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))} \leq C_T\sqrt{k/N}.$$

$\Rightarrow$ A quantitative propagation of chaos in terms of the TV-distance requires moment control on $Z_T^{N,\infty}$.

Local control for $Z_T^{N,\infty}$ for the prototypical McKean-Vlasov model:

$$Z_T^{N,\infty} = \exp\left\{-\sum_{i=1}^N \int_0^T \Delta B_t^{i,N}\cdot dW_t^i - \frac{1}{2}\int_0^T \sum_{i=1}^N \left|\Delta B_t^{i,N}\right|^2 dt\right\},$$

$$\Delta B_t^{i,N} = \sigma^{-1}\left(\frac{1}{N}\sum_{j=1}^N b(X_t^{i,\infty} - X_t^{j,\infty}) - \int b(X_t^{i,\infty} - y)\mu(t,dy)\right).$$

Proposition 6 (Local control for $Z^{N,\infty}$, J' 2019, [?])

For all $0 < T_0 < T < \infty$, $0 < \kappa < \infty$,

$$\sup_N \mathbb{E}\left[\exp\left\{\kappa\sum_{i=1}^N \int_{T_0}^{T_0+\delta} \Delta B_t^{i,N}\cdot dW_t^i\right\}\right] < \infty,$$

whenever $\delta < \left(16^2\kappa T d^2\|\sigma^{-1}b\|^2_{L^\infty}\right)^{-1}$.

Elements of proof (inspired by Krylov and Rockner 2005): Exponential's expansion:

$$\mathbb{E}\left[\exp\left\{\kappa\sum_{i=1}^N \int_{T_0}^{T_0+\delta} \Delta B_t^{i,N}\cdot dW_t^i\right\}\right] = \sum_l \frac{\kappa^l}{l!}\mathbb{E}\left[\left(\sum_{i=1}^N \int_{T_0}^{T_0+\delta} \Delta B_t^{i,N}\cdot dW_t^i\right)^l\right]$$

$$\leq 2\sum_l \frac{(\kappa)^{2l}}{(2l)!}\left(1 + \mathbb{E}\left[\left(\sum_{i=1}^N \int_{T_0}^{T_0+\delta} \Delta B_t^{i,N}\cdot dW_t^i\right)^{2l}\right]\right).$$

$$
\mathbb{E}\left[\left(\sum_{i=1}^{N}\int_{T_0}^{T_0+\delta}\Delta B_t^{i,N}\cdot dW_t^i\right)^{2l}\right]
$$

$$
\leq 2^{2l} l^l \mathbb{E}\left[\left(\sum_{i=1}^{N}\int_{T_0}^{T_0+\delta}\left|\Delta B_t^{i,N}\right|^2 dt\right)^{l}\right]
$$

$$
\leq 2^{2l} l^l N^l \delta^{l-1}\mathbb{E}\left[\int_{T_0}^{T_0+\delta}\left|\frac{1}{N}\sum_{j=1}^{N}\sigma^{-1}\left(b(X_t^{i,\infty}-X_t^{j,\infty})-\int b(X_t^{i,\infty}-y)\mu(t,dy)\right)\right|^{2l} dt\right].
$$

Corollary 7 (Bougeron *et al.* [2])

Let $X_1, X_2, \cdots, X_n$ be a sequence of i.i.d. r.v.s' such that a.s. $|X_1| \leq \overline{m} < \infty$. Then

$$
\mathbb{E}[\left(\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i])\right)^{2q}] \leq q!(2n\overline{m}^2)^q.
$$

Owing to the $\delta < \overline{\delta}$ for $\overline{\delta} = \left(16^2\kappa T d^2\|\sigma^{-1}b\|_{L^\infty}^2\right)^{-1}$. We recover a series of the form

$$
\sum_{l=1}^{\infty} a_l,\ a_l = \frac{(l)^l l! C^l \delta^l}{(2l)!},
$$

for C depending only on $\kappa\ T, \|\sigma^{-1}b\|_{L^\infty}$. Choosing δ small enough ensures that the sum is finite.

Theorem 8 ([?])

Assume that σ *is definite positive and* $b : (0, T) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *is a bounded Borel function. Then*

$$\|P^{k,N} - P^{k,\infty}\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^{kd}))} \leq \frac{C_{k,d,T,\|\sigma^{-1}b\|_{L^\infty}}}{\sqrt{N}}.$$

Elements of proof: Decomposing the interval $[0, T]$ into $\cup_{m=0}^{M-1}[m\delta,(m+1)\delta \wedge T]$, $M = \lfloor T/\delta \rfloor$: and, for each $0 \leq m \leq M$, define the family of processes $(Y_t^{1,N,m,\infty})_{t\leq T}$, , $(Y_t^{N,N,m,\infty})_{t\leq T}$ as

$$Y_t^{i,N,m} = \begin{cases} X_0^i + \int_0^t \int b(Y_s^{i,N,m} - y)\mu^i(s,dy)\,ds + \sigma \widehat{W}_t^i,\ 0 \leq t \leq m\delta,\ \mu^i(t) = \mathcal{L}(Y_t^i), \\ Y_{m\delta}^{i,N,m} + \frac{1}{N}\sum_{j=1}^N \int_{m\delta}^t b(Y_s^{i,N,m} - Y_s^{j,N,m})\,ds + \sigma(W_t^i - W_{\delta m}^i),\ m\delta < t \leq T \wedge \delta M. \end{cases}$$

This system corresponds to a family of N-independent McKean-Vlasov, that progressively interact which others on a time interval of the form $(m\delta, T]$.

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$$Y_t^{i,N,m} = \begin{cases} X_0^i + \int_0^t \int b(X_s^{i,\infty} - y)\mu(s,dy)\,ds + \sigma \widehat{W}_t^i,\ 0 \leq t \leq m\delta, \\ X_{m\delta}^{i,\infty} + \frac{1}{N}\sum_{j=1}^N \int_{m\delta}^t b(X_s^{i,\infty} - X_s^{j,\infty})\,ds + \sigma(W_t^i - W_{\delta m}^i),\ m\delta < t \leq T \wedge \delta M. \end{cases}$$

This system corresponds to a family of N-independent McKean-Vlasov, that progressively interact which others on a time interval of the form $(m\delta, T]$.

By the triangular inequality

$$\|P^{1,\infty}-P^{1,N}\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))}\leq\sum_{m=0}^{M}\|P^{1,m+1,N}-P^{1,m,N}\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))}$$

where $P^{1,m+1,N}$ is the probability measure generated by $(Y_t^{i,N,m})_{0\leq t\leq T}$. The exponential martingale which enables to change $Y^{i,m+1,N}$ into $Y^{i,m,N}$ is given by

$$\exp\left\{\sum_{i=1}^{N}\int_{m\delta}^{(m+1)\delta\wedge T}\left(\int b(Y_s^{i,N,m+1}-y)\mu^i(s,dy)-\frac{1}{N}\sum_{j=1}^{N}b(Y_s^{i,N,m}-Y_s^{j,N,m})\right)dW_s^i\right.$$
$$\left.-\frac{1}{2}\sum_{i=1}^{N}\int_{m\delta}^{(m+1)\delta\wedge T}\left|\int b(Y_s^{i,N,m+1}-y)\mu^i(s,dy)-\frac{1}{N}\sum_{j=1}^{N}b(Y_s^{i,N,m}-Y_s^{j,N,m})\right|^2ds\right\}$$
$$=\exp\left\{-\sum_{i=1}^{N}\int_{m\delta}^{(m+1)\delta\wedge T}\Delta B_t^{i,N}\cdot dW_t^i-\frac{1}{2}\int_{m\delta}^{(m+1)\delta\wedge T}\sum_{i=1}^{N}\left|\Delta B_t^{i,N}\right|^2dt\right\}.$$

Each component $\|P^{1,m+1,N}-P^{1,m,N}\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))}$ can then be controlled in terms of $C/\sqrt{N}$.

Remark: The proof only relies on the control of the exponential martingale which enables to pass from N-independent copies of the McKean-Vlasov models to the N-interacting particle systems and since this control is based on moment control of the difference $\triangle B_t^{i,N}$ given by some concentration inequality/sub-gaussian moment controls $\Rightarrow$ Room for improvement.

- Langevin McKean-Vlasov models:

Theorem 9 ([?])

Assume that $b : \mathbb{R}^d \to \mathbb{R}^d$ *is a bounded Borel function,* σ *definite positive. Then*

$$\|\mathcal{L}(((Y_t^{i,N}, V_t^{i,N});\, t \leq T)) - \mathcal{L}(((Y_t^{i,\infty}, V_t^{i,\infty});\, t \leq T))\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^{kd}))} \leq C/\sqrt{N},$$

where

$$\begin{cases} dY_t^{i,N} = V_t^{i,N}\, dt,\ 1 \leq i \leq N, \\ dV_t^{i,N} = \Big(\frac{1}{N}\sum_{j=1}^{N} b(Y_t^{i,N} - Y_t^{j,N}, V_t^{i,N} - V_t^{j,N})\Big)\, dt + \sigma dW_t^i, \end{cases}$$

$$\begin{cases} dY_t^{i,\infty} = V_t^{i,\infty}\, dt, \\ dV_t^{i,\infty} = \Big(\int b(Y_t^{i,\infty} - y, V_t^{i,\infty} - v)\, \mu(t, dy, dv)\Big)\, dt + \sigma dW_t^i. \end{cases}$$

Rate of convergence for path-dependent McKean-Vlasov model: Consider the systems:

$$dX_t^{i,\infty} = B\big(t, (X_r^{i,\infty})_{0\le r\le t}, \mathcal{L}((X_r^{i,\infty})_{0\le r\le t})\big)\, dt + A\big(t, (X_r^{i,\infty})_{0\le r\le t}\big)\, dW_t^i,$$

$$dX_t^{i,N} = B\big(t, (X_r^{i,N})_{0\le r\le t}, \frac{1}{N}\sum_{i=1}^{N} \delta_{\{(X_r^{i,N})_{0\le r\le t}\}}\big)\, dt + A\big(t, (X_r^{i,N})_{0\le r\le t}\big)\, dW_t^i,$$

under the same assumptions of Lacker 2018. Assume further the centering

$$\mathbb{E}[B\big(t, (X_r^{i,\infty})_{0\le r\le t}, \frac{1}{N}\sum_{i=1}^{N} \delta_{\{(X_r^{i,\infty})_{0\le r\le t}\}}\big)] = B\big(t, (X_r^{i,\infty})_{0\le r\le t}, \mathcal{L}((X_r^{i,\infty})_{0\le r\le t})\big)$$

Such condition ensures that

$$\mathbb{E}\left[\left(\int_0^T \left|(A^{-1}B)\big(t, (X_s^{1,\infty})_{0\le s\le t}, \Gamma_t^N\big) - (A^{-1}B)\big(t, (X_r^{1,N})_{0\le r\le t}, \mathcal{L}((X_r^{1,N})_{0\le r\le t})\big)\right|^2 dt\right)^p\right] \le \frac{p!\beta^p}{N^p}$$

and next

$$\|\mathcal{L}((X_t^{1,N},\dots,X_t^{k,N};\ t\le T) - \mathcal{L}((X_t^{1,\infty},\dots,X_t^{k,\infty};\ t\le T)\| - TV = \mathcal{O}(\sqrt{1/N}).$$

Note: More general cases would require more structural assumptions on B.

Conditional McKean Lagrangian model:

$$(*)\ \begin{cases} dX_t = U_t\,dt, \\ dU_t = \mathbb{E}\left[b(U_t)\,|\,X_t\right]\,dt + \sigma dW_t, \\ (X_0, U_0) \sim \mu_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d). \end{cases}$$

Theorem 10 (Bossy, J. and Talay 2011)

Assume that $b : \mathbb{R}^d \to \mathbb{R}^d$ is bounded continuous, $\sigma \neq 0$ and μ_0 admits a Lebesgue density. Then there exists a unique solution to $(*)$ and, as $N \to +\infty$ and $\epsilon \to 0^+$, the particle system

$$\begin{cases} X_t^{i,\epsilon,N} = X_0^i + \int_0^t U_s^{i,\epsilon,N}\,ds, \\ U_t^{i,\epsilon,N} = U_0^i + \int_0^t \dfrac{\frac{1}{N}\sum_{j=1}^N b(U_s^{j,\epsilon,N})\phi_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N})}{\frac{1}{N}\sum_{j=1}^N \phi_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N}) + \epsilon}\,ds + \sigma W_t^i, \\ \{\phi_\epsilon\} \text{ family of smooth mollifiers,} \end{cases}$$

converges in the sense of propagation of chaos toward $(*)$.

Rate of convergence: (joint work with S. Menozzi)
Intermediate smoothed dynamic:

$$X_t^\epsilon = X_0 + \int_0^t U_s^\epsilon \, ds,\; U_t^\epsilon = U_0 + \int_0^t \frac{\mathbb{E}[b(\tilde{U}_s^\epsilon)\phi_\epsilon(x-\tilde{X}_s^\epsilon)]}{\mathbb{E}[\phi_\epsilon(x-\tilde{X}_s^\epsilon)]+\epsilon}\Big|_{x=X_s^\epsilon} ds + \sigma W_t,$$

for $(\tilde{X}_t, \tilde{U}_t)$ independent copy of (X_t, U_t).
Error decomposition: For $P^{i,\epsilon,N}$, $P^{\epsilon,\infty}$ and P^∞ the respective law on $\mathcal{C}([0,T];\mathbb{R}^d)$ of $(X^{i,\epsilon,N}, U^{i,\epsilon,N})$, (X^ϵ, U^ϵ) and (X, U),

$$\begin{aligned}&\|P^{i,\epsilon,N} - P^\infty\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))}\\ \leq\; & \underbrace{\|P^{i,\epsilon,N} - P^{\epsilon,\infty}\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))}}_{\sim C(\epsilon)/\sqrt{N} \text{ due to the singularity of the conditional expectation}}\\ &+ \underbrace{\|P^{\epsilon,\infty} - P^\infty\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))}}_{\text{requires a sharp estimate on the regularity of the marginal dist. of } P, P^\epsilon} .\end{aligned}$$

For the estimate on $\|P^{\epsilon,\infty} - P^\infty\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))}$: As

$$\|P^{\epsilon,\infty} - P^\infty\|_{TV,\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))} \leq \sqrt{2H(P^{\epsilon,\infty} \,|\, P^\infty)}$$

$$\leq \sqrt{2\mathbb{E}\left[\int_0^T \left|\frac{\mathbb{E}[b(\tilde{U}_s^\epsilon)\phi_\epsilon(x-X_s^\epsilon)]}{\mathbb{E}[\phi_\epsilon(x-X_s^\epsilon)]+\epsilon}\Big|_{x=X_s} - \mathbb{E}[b(U_s)\,|\,X_s]\right|^2 ds\right]}$$

the control of this expression relies on density estimates of $\mathcal{L}(X_t)$ and $\mathcal{L}(X_t^\epsilon)$.

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