McKean–Vlasov SDEs with Common Noise

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Why Condition 2?

Imagine that one is solving a stochastic equation

$$\Gamma(Y,Z)=0, \quad Y\sim\nu.$$

Y is the stochastic input with determined distribution ν and Z is the solution/output. Often, one seeks to solve the above by instead considering a mollified equation $\Gamma^n(Y,Z)=0$ such that "$\Gamma^n\to\Gamma$" and $\forall n$ the equation is *strongly* solvable; there is a measurable function F^n such that $Z^n := F^n(Y)$ is a solution. Then, passing to the limit in some sense "$\Gamma^n(Y,Z^n)\to\Gamma(Y,Z)$".

In the case of compactness arguments (weak existence) one proves the weak convergence of a subsequence of the joint distributions of approximate solutions (Y,Z^n) and represents the solutions on a another probability space $(\bar\Omega,\bar{\mathcal{F}},\bar{\mathbb{P}})$ such that $(\bar Y^n,\bar Z^n)\to(\bar Y,\bar Z)$ surely. Since $(\bar Y^n,\bar Z^n)$ have the same distribution as (Y,Z^n), $F^n(\bar Y^n)=\bar Z^n$. Therefore $\bar Z$ is the pointwise limit of $\bar Y^n$ measurable functions, but unfortunately, $\bar Y^n$ varies along the same limit, and one cannot conclude that there is a measurable function F such that $\bar Z = F(\bar Y)$. So, in the case of McKean-Vlasov with Common noise, one can show $\bar\mu^n_t = \mathscr{L}(\bar X^n_{\cdot\wedge t}|\mathcal{F}^{\bar B^n}_t)$, yet $\bar\mu_t=\lim_n\bar\mu^n_t$ cannot be claimed to be $\mathcal{F}^{\bar B}_t$ measurable. Seeking to identify the connection of the limiting random measure $\bar\mu$ to $\bar X$, the relaxation $\bar\mu_t=\mathscr{L}(\bar X_{\cdot\wedge t}|\mathcal{F}^{\bar B,\bar\mu}_t)$ is made.

Existence Proof Rationale

The existence result follows from a standard method; a combination of the Arzelà-Ascoli Theorem, Skorokhod Representation and Convergence Lemmas for Stochastic integrals amongst other tools. In addition, some weak convergence arguments are employed to handle the extra compatibility conditions, and these motivate demanding $\mu_t=\mathscr{L}(X_{\cdot\wedge t}|\mathcal{F}^{B,\mu}_t)$ rather than $\mathscr{L}(X_t|\mathcal{F}^{B,\mu}_t)$ as the former carries more information about the dependence structure between X and (B,μ). To connect the flow of measures μ_t to the SPDE (2), a Stochastic Fubini theorem is used to handle the fact that μ carries some randomness external to B. Indeed, one needs to be able to show that for $\varphi\in C_0^\infty$:

$$\mathbb{E}\left[\int_0^t \nabla\varphi(X_s)\rho(s,X_s,\mu_s)\,dB_s\middle|\mathcal{F}^{B,\mu}_t\right] = \int_0^t \mathbb{E}[\nabla\varphi(X_s)\rho(s,X_s,\mu_s)|\mathcal{F}^{B,\mu}_s]\,dB_s.$$

The compatibility assumption enables such an equality.

This poster concerns the McKean-Vlasov Stochastic Differential Equation with Common Noise, the $N\to\infty$ limit of N-particle systems satisfying, for $i\in\{1,\dots,N\}$, $t\in I:=[0,\infty)$,

$$dX^i_t = b(t,X^i_t,\mu^N_t)dt+\sigma(t,X^i_t,\mu^N_t)dW^i_t+\rho(t,X^i_t,\mu^N_t)dB_t \qquad \mu^N_t := \frac{1}{N}\sum_{j=1}^N \delta_{X^j_{\cdot\wedge t}}$$

$\downarrow$ as the number of particles $N\to\infty$

$$dX_t = b(t,X_t,\mathscr{L}(X_{\cdot\wedge t}|\mathcal{F}^B_t))dt+\sigma(t,X_t,\mathscr{L}(X_{\cdot\wedge t}|\mathcal{F}^B_t))dW_t+\rho(t,X_t,\mathscr{L}(X_{\cdot\wedge t}|\mathcal{F}^B_t))dB_t$$

B is the common noise, W^i are the private noises, all mutually independent and independent of the common noise and $\mathscr{L}(X^i_{\cdot\wedge t}|\mathcal{F}^B)$ denotes the regular conditional distribution of $X^i_{\cdot\wedge t}$ given $\mathcal{F}^B_t$. These equations have been studied in numerous contexts by, for example, Kurtz and Xiong [4] and very recently by Coghi and Gess [2] to study a non-linear SPDE of measure, and also Carmona, Delarue and Lacker [1] for Mean Field Games. If one wishes to depart from the Global Lipschitzean framework (as in [1]) and establish an existence and uniqueness theory, then numerous issues arise.

Definition of Solutions of the McKean–Vlasov SDE with Common Noise.

A weak solution consists of a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ equipped with $\mathbb{F}$ Brownian motions B and W and $\mathcal{F}_0$ measurable initial condition ξ, all mutually independent, along with $\mathbb{F}$ adapted processes X and μ that are $\mathbb{R}^{d_X}$ and $\mathcal{P}(C(I;\mathbb{R}^{d_X}))$ valued respectively, satisfying the following conditions:

1. $\int_0^t |b(s,X_s,\mu_s))|+|\sigma(s,X_s,\mu_s))|^2+|\rho(s,X_s,\mu_s))|^2\,ds<\infty$ $\mathbb{P}$-a.s. for all $t\in I$
2. $\mu_t=\mathscr{L}(X_{\cdot\wedge t}|\mathcal{F}^{B,\mu}_t)$ for all $t\in I$
3. X is *compatible* with (B,μ) in the sense that $\mathcal{F}^X_t$ is independent of $\mathcal{F}^{B,\mu}_\infty$ given $\mathcal{F}^{B,\mu}_t$ for all $t\in I$. Also, $(W,\xi)\perp\!\!\!\perp(B,\mu)$ and (X,μ) is compatible with (B,W,ξ)
4. $\mathbb{P}$-a.s. for all $t\in I$,

$$X_t=\xi+\int_0^t b(s,X_s,\mu_s)\,ds+\int_0^t\sigma(s,X_s,\mu_s)\,dW_s+\int_0^t\rho(s,X_s,\mu_s)\,dB_s. \tag{1}$$

If X,μ are $\{\mathcal{F}^{B,W,\xi}_t\}_t := \{\sigma(\xi,B_s,W_s : s\le t)\}_t$ adapted, then the solution is *strong*.

A brief explanation of the seemingly unusual conditions 2. and 3. would be that compatibility is usually hidden (in the case of non-common noise McKean-Vlasov and Standard SDE theory) within the claim that the solution is adapted to some filtration for which the input process W is adapted and in this setting, compatibility is needed to connect solutions to (1) to solutions of the Stochastic Fokker Planck equation (2). The relaxation that μ is $\mathscr{L}(X|\mathcal{F}^{B,\mu})$ opposed to simply $\mathscr{L}(X|\mathcal{F}^B)$ comes from the fact that one expects to obtain weak solutions via compactness arguments and measurability is unstable under weak limits.

An Existence Result

Theorem 1. *Let assumption 4 (top right) hold. Then for $I=[0,T]$, there exists a weak solution to the McKean-Vlasov SDE with Common Noise and the measure flow μ projected onto time marginals $\tilde\mu_t := \mu_t\circ\psi_t^{-1}$ is a weak solution to the SPDE (2).*

A case studied by Mishura and Veretennikov in [5] for the normal McKean-Vlasov setting can be adapted to the setting with a common noise:

Assumption 2. The coefficients b,σ and ρ are measurable, (σ,ρ) do not depend on the measure argument and are such that there exists a unique strong solution to the driftless SDE:

$$dX^0_t=\sigma(t,X^0_t)dW_t+\rho(t,X^0_t)dB_t.$$

Also, σ is non-degenerate, invertible and $\sigma^{-1}b$ is bounded and Lipschitz continuous in the measure component with respect to the total variation distance, i.e. there is a constant c_{TV} such that,

$$|\sigma(t,x)^{-1}b(t,x,\mu)-\sigma(t,x)^{-1}b(t,x,\nu)|\le c_{\mathrm{TV}}d_{\mathrm{TV}}(\mu,\nu).$$

Joint Weak Uniqueness

Theorem 3. *Under assumption 2, the McKean-Vlasov Equation with Common Noise satisfies uniqueness of joint distribution.*

Coefficients and Initial Condition

Assumption 4. Functions b, σ and ρ depend on measure only through the time marginals, i.e $b(t,X_t,\mathscr{L}(X_{\cdot\wedge t}|\mathcal{F}^{B,\mu}_t)) = \tilde b(t,X_t,\mathscr{L}(X_t|\mathcal{F}^{B,\mu}_t))$. Further, they are bounded and jointly continuous in the last two arguments in the following sense: if $(x_n\to x,\mu_n\xrightarrow{w}\mu)$ as $n\to\infty$ then $(b,\sigma,\rho)(t,x_n,\mu_n)\to(b,\sigma,\rho)(t,x,\mu)$ as $n\to\infty$. The initial data has, for some fixed $p\in[1,\infty]$, $||\xi||_p<\infty$.

A Non-Linear SPDE

A probabilistically weak solution to the SPDE (2) is a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ equipped with an $\mathbb{F}$ Brownian motion B with $\mathbb{F}$ adapted $\mathcal{P}(\mathbb{R}^{d_X})$ valued process ν satisfying, for all $t\in I$ and for all test functions $\varphi\in C_0^\infty(\mathbb{R}^{d_X})$

$$\begin{aligned}\langle\nu_t,\varphi\rangle &= \langle\nu_0,\varphi\rangle+\int_0^t\langle\nu_s,L\varphi(s,\cdot,\nu_s)\rangle\,ds \\ &\quad+\int_0^t\langle\nu_s,\nabla\varphi\tilde\rho(s,\cdot,\nu_s)\rangle\,dB_s \ \ \mathbb{P}\text{-a.s.}\end{aligned} \tag{2}$$

If ν is $\{\mathcal{F}^B_s\}_t$ adapted, then the solution is *strong*.

Theorem 3 Proof Rationale

In order to compare the distributions of the two solutions, one needs method of comparison that leverages the fact that $\mu=\mathscr{L}(X|\mathcal{F}^{B,\mu})$. However, for arbitrary couplings putting two solutions on the same probability space, the dependence structure between the two random measures μ^1 and μ^2 obstructs the ability to estimate, say $\mathbb{E}d_{\mathrm{TV}}(\mu^1,\mu^2)\le\mathbb{P}[X^1\ne X^2]$. Employing a coupling that fixes the underlying randomness of both μ^1 and μ^2 to be the same, it becomes possible to prove distributional uniqueness by representing the two solutions by Girsanov Transformations of the unique process X^0.

Lyapunov Criteria for Existence and Uniqueness

It is also possible to extend the results of a previous work by H., Šiška and Szpruch [3] to the common noise setting. These existence and uniqueness arguments leverage the existence of Lyapunov functions for the equation (1).

References

[1] René Carmona, François Delarue, and Daniel Lacker. Mean field games with common noise. *The Annals of Probability*, 2016.

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[5] Yuliya S Mishura and Alexander Yu Veretennikov. Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations. *arXiv:1603.02212*, 2016.