Uniform in time Estimates for the weak error of the Euler method for SDEs

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Based on joint work with D. Crisan and M. Ottobre

Introduction

We are interested in approximating stochastic processes which are solutions of the SDE:

$$dX_t = U_0(X_t)dt + \sqrt{2}\sum_{i=1}^{d} V_i(X_t)dW_t^i.$$

The Euler approximation is given by

$$Y^\delta_{t_{n+1}} = Y^\delta_{t_n} + U_0(Y^\delta_{t_n})\delta + \sqrt{2}\sum_{k=1}^{d} V_k(Y^\delta_{t_n})\Delta B^k_{t_n}$$

where $t_n = n\delta$ and $\Delta B_{t_n} = B_{t_{n+1}} - B_{t_n}$.
The weak error can in general be estimated, see [3], by

$$\sup_{t_n\le T}\left|\mathbb{E}[\varphi(X_{t_n})] - \mathbb{E}[\varphi(Y^\delta_{t_n})]\right| \le C(T,\varphi)\delta$$

The aim is to find conditions under which $C(T,\varphi)$ can by taken independent of T. These will be in terms of the semigroup $\mathcal{P}_t$ defined by

$$\mathcal{P}_t f(x) := \mathbb{E}[f(X_t)|X_0 = x].$$

Assumptions

1. The SDE is uniformly elliptic.
2. The vector fields $U_0, V_1, \dots, V_d$ are smooth and Lipschitz.
3. There exist constants $\lambda_0, C > 0$ such that for all $f \in C^4_b(\mathbb{R}^N)$ we have

$$\sum_{k=1}^{4}\sum_{i_1,\dots,i_k=1}^{d} |\partial_{i_1,\dots,i_k}\mathcal{P}_t f(x)| \le Ce^{-\lambda_0 t}\|f\|_{C^4_b}.$$

4. For δ sufficiently small

$$\sup_{n\in\mathbb{N}} \mathbb{E}[(Y^\delta_{t_n})^4] < \infty.$$

Main Theorem

Let the above assumptions hold. Then the weak error of the Euler approximation $\{Y^\delta_t\}_{t\ge0}$ converges to 0, uniformly in time, as $\delta \to 0$; that is, there exists some constant K depending only on λ_0, d and N such that for all $\varphi \in C^4_b(\mathbb{R}^N)$ and $\delta > 0$ we have

$$\sup_{t\ge0}|\mathbb{E}[\varphi(X_t)] - \mathbb{E}[\varphi(Y^\delta_t)]| \le K\delta\|\varphi\|_{C^4_b}.$$

Obtuse Angle Condition

Suppose for all $i \in \{1,\dots,d\}, \xi, x \in \mathbb{R}^N$

$$\xi^T[V_i, V_0](x)V_i(x)^T\xi \le -\lambda_0|\xi^T V_i(x)|^2.$$

Here V_0 is the Stratonovich form of the drift coefficient, i.e.

$$V_0(x) = U_0(x) - \sum_{k=1}^{d}\sum_{j=1}^{N} V_k^j(x)\partial_j V_k^i(x).$$

This implies, see [2], that

$$|V_i(x)\cdot\nabla\mathcal{P}_t f(x)| \le e^{-\lambda t}\|f\|_{C^1_b(\mathbb{R}^N)}.$$

Examples

Consider the two-dimensional SDE

$$dX^1_t = X^1_t dt$$
$$dX^2_t = \sqrt{2}X^1_t dB_t$$

In this case the Obtuse Angle Condition is satisfied, hence the derivatives decay exponentially. However neither the solution X_t nor the approximation $Y^\delta_{t_n}$ are tight. Hence exponentially decaying derivatives doesn't imply bounds on the moments of the process.

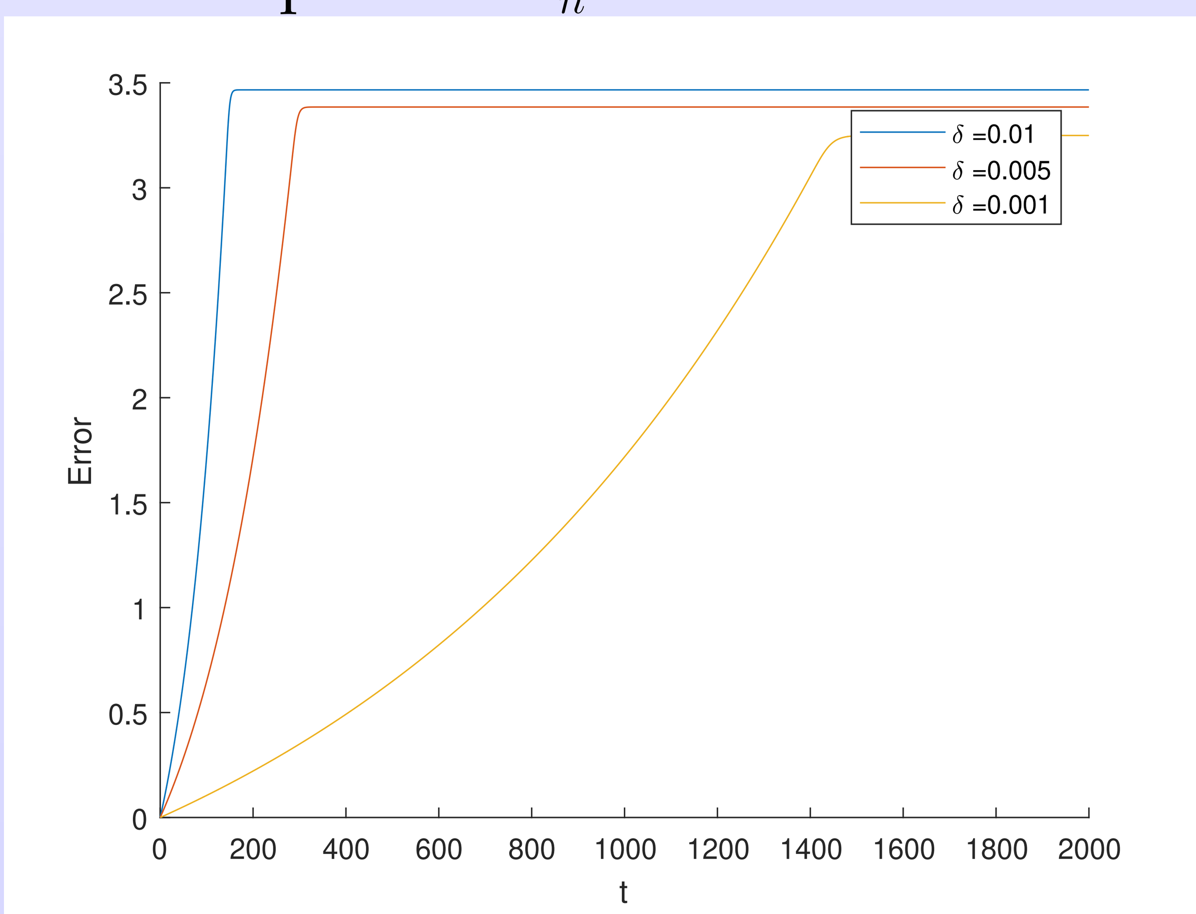
Moreover, tightness of the process does not imply tightness of the approximation for example consider

$$dX_t = (-X_t^3 - X_t)dt + \sqrt{2}dB_t.$$

It is shown in [4] that if $\mathbb{E}[(Y^\delta_0)^2] \ge 4 + 4/\delta^2$ then $\mathbb{E}[(Y^\delta_{t_n})^2] \to \infty$ as $n \to \infty$. However X_t has second moment bounded uniformly in time.
Conversely, bounded moments does not imply that the weak error converges to zero uniformly in time nor that the derivatives of the semigroup converge to zero exponentially fast. Consider the two dimensional ODE

$$\begin{cases} \frac{d}{dt}X^1_t = \left(-X^2_t + \Psi(X_t)X^1_t\right)dt \\ \frac{d}{dt}X^2_t = \left(X^1_t + \Psi(X_t)X^2_t\right)dt \end{cases} \quad (1)$$

where $\Psi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth bounded function such that $\Psi(x) = 0$ if $|x| < 2$ and $\Psi(x) = -1$ if $|x| > 3$. In this case the solution X_t lives on the circle of radius 1 and centre 0, however if $R^\delta_n := |Y^\delta_{t_n}|$ then R^δ_n converges to a number greater than 2. Below is a plot of $R^\delta_n - 1$ for different choices of δ.



References

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