Supercooled Stefan problem in 1d: solutions beyond blow-up

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Joint work S. Nadtochiy (Chicago) and M. Shkolnikov (Princeton)

Part I. Motivation and Main results

Supercooling

- Liquid water may exist in metastable state under $0°C$
  - small perturbations or contact with ice $\Rightarrow$ solidification
  - see the video ( )
  - formulated by Stefan [in late 19's] and Brillouin [early 20's]
- Description of the front of ice during solidification
  - PDE point of view: heat equation with free boundary

    $\rightsquigarrow$ ill-posed in classical sense $\leadsto$ speed of propagation may become infinite

    $\rightsquigarrow$ description up to the emergence of a singularity in the propagation of the front [Fasano et al., DiBenedetto et al., 80's]
  - new interest in probability for several years: maths finance, neurosciences with singular mean field interaction
  - purpose of the talk: go beyond singularities

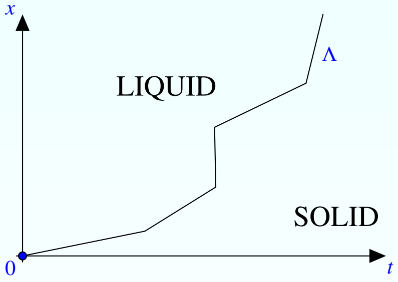
    $\rightsquigarrow$ use a probabilistic approach of the problem

PDE formulation

- Work in dimension 1
- Denote by Λ_t the position of the front at time t

  $\rightsquigarrow$ ice below Λ_t and liquid above Λ_t

  - $\Lambda_0 = 0$



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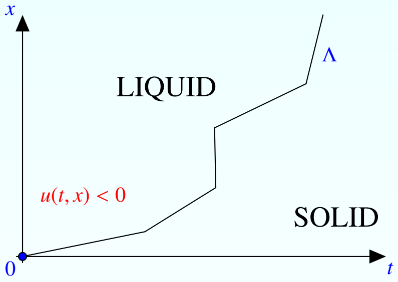

  - initial profile for the temperature $u(0, x)$, $x \geq 0$
  - inside liquid phase $\rightsquigarrow$ heat equation for $u(t, x)$, $x > \Lambda_t$

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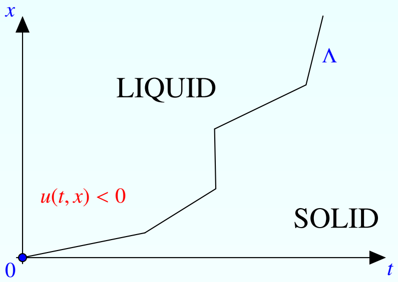

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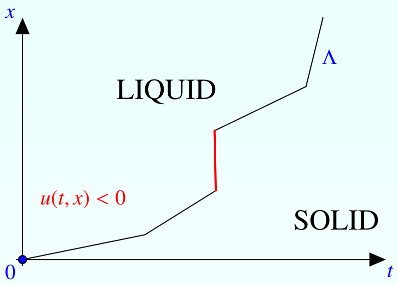

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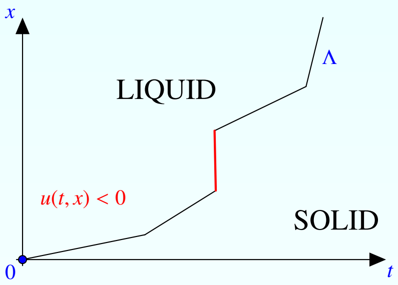

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- Regard $-u(t, \cdot)$ as a density (with properly normalized initial condition)
- Provide particle description of the dynamics ($\alpha = 1$)

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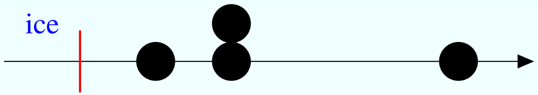

∘ assume N particles and distance between sites is $\frac{1}{N}$

∘ black particles jump at exponential times, with 1/2 probability to the left and 1/2 to the right

$\rightsquigarrow$ as long as they do not touch the front!

$\rightsquigarrow$ when one particle hits the front, the front moves forward!

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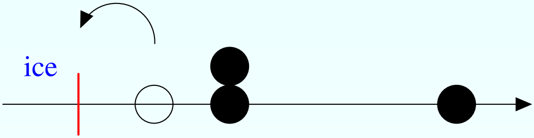

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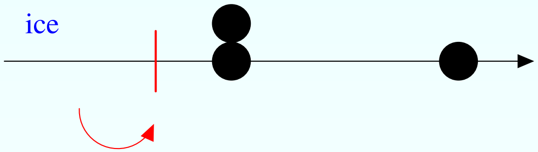

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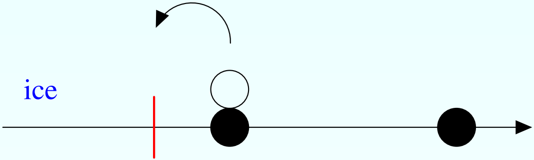

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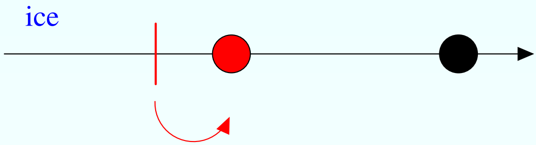

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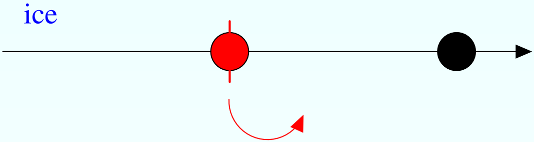

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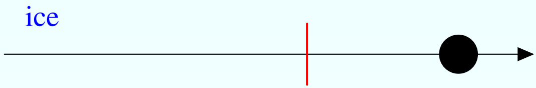

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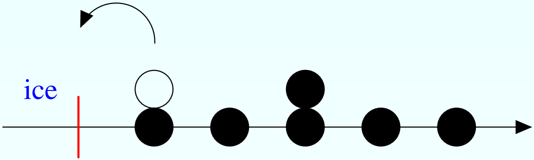

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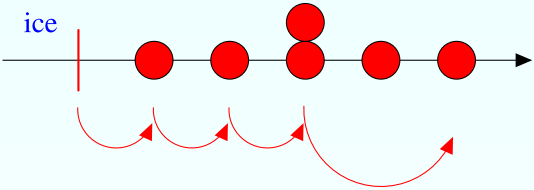

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Continuous version [D. et al., Hambly et al., N. S.]

• May easily replace the discrete dynamics by continuous dynamics inside the liquid phase

• N particles

∘ evolve like independent Brownian motions before one them touches the front

∘ each time one particle is absorbed by the front, the front receives an upward kick of size α/N

$\rightsquigarrow$ particle $\sharp i \in \{1, \cdots, N\}$

$$X_t^i = X_0^i + B_t^i, \qquad t \le \tau^i = \inf\{s \ge 0 : X_s^i \le \Lambda_s\}$$

$$\Lambda_t = \frac{\alpha}{N} \sum_{j=1}^{N} \mathbf{1}_{\{\tau^j \le t\}}$$

∘ $X_0^1, \cdots, X_0^N \perp\!\!\!\perp$ initial conditions, $B^1, \cdots, B^N \perp\!\!\!\perp$ Brownian motions

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Probabilistic formulation

- (Formal) mean field limit $\rightsquigarrow$ provide the dynamics of one typical particle within the population

$$X_t = X_0 + B_t - \alpha \mathbb{P}(\tau \le t), \quad t \le \tau = \inf\{s \ge 0 : X_s \le 0\}$$

$\rightsquigarrow$ attention: here, focus on the distance from the particle to the front

  - front is here given by $\Lambda_t = \alpha \mathbb{P}(\tau \le t)$

- Formal connection with Stefan problem

$$u(t, x + \Lambda_t) = \underbrace{-\frac{d}{dx}\mathbb{P}(X_t \in [x, x + dx],\ t < \tau)}_{p(t,x)}$$

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$$\dot{\Lambda}_t = \frac{\alpha}{2}\partial_x p(t, 0) \quad \text{(hitting time } dX_t = -\dot{\Lambda}_t dt + dB_t, t)$$

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$$\int_{\Lambda_t}^{\infty} \Big(\partial_t u(t,x) - \frac{1}{2}\partial_x^2 u(t,x)\Big)dx = 0 \Rightarrow -\frac{1}{\alpha}\dot{\Lambda}_t = \frac{1}{2}\partial_x u(t,\Lambda_t)$$

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Physical solution

- Possible jumps of Λ (Λ taken càd-làg)

$$\Lambda_t - \Lambda_{t-} = \frac{\alpha}{N} \sum_{j=1}^{N} \mathbf{1}_{\{\tau^j = t\}}$$

  - too weak description of the jumps

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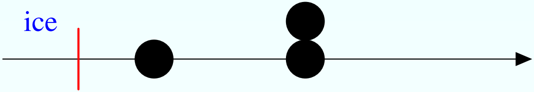

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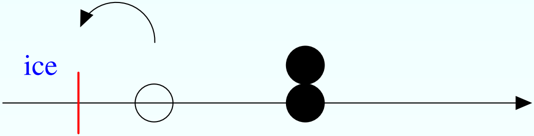

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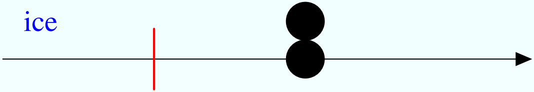

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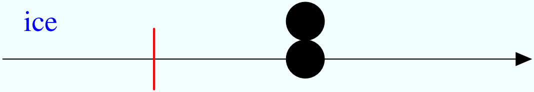

$\rightsquigarrow$ $\Lambda_t - \Lambda_{t-} = 1/N$

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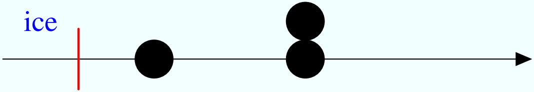

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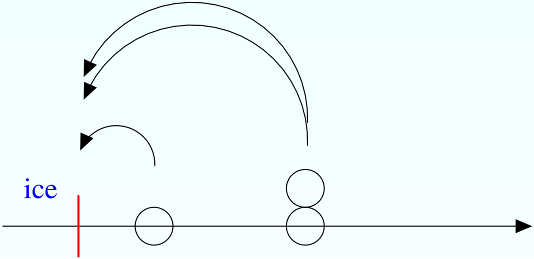

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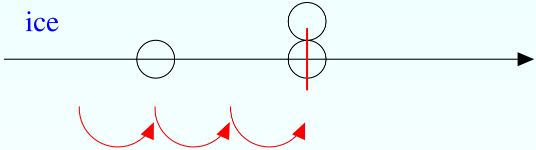

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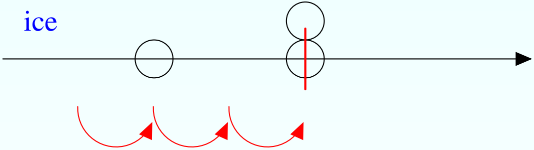

  - ○ need condition to force jumps to be ordered

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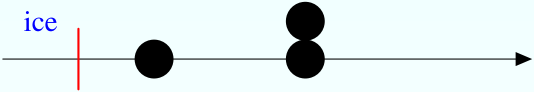

- To get jump $\geq x$ at time t, need

$$\underbrace{\frac{\alpha}{N} \sum_{j=1}^{N} \mathbf{1}_{X^j_{t-} \in (0,x]}}_{\text{contribution to the jump of particles } \leq x} \qquad \geq x$$

Physical solution

- Possible jumps of Λ
  - original description of the jumps is too weak
- To get jump $\geq x$ at time t, need the mass

$$\underbrace{\frac{\alpha}{N}\sum_{j=1}^{N}\mathbf{1}_{X^j_{t-}\in(0,x]}}_{\text{contribution to the jump of particles} \leq x} \geq x$$

- Write the mean field equation in the form

$$X_t = X_0 + B_t - \alpha\mathbb{P}(\tau \leq t), \quad t \leq \tau = \inf\{s \geq 0 : X_s \leq 0\}$$

$$\rightsquigarrow \Lambda_t = \alpha\mathbb{P}(\tau \leq t)$$

  - require

$$\Lambda_t - \Lambda_{t-} = \inf\Big\{x \geq 0 : \alpha\mathbb{P}(X_{t-} \in (0,x]) < x\Big\}$$

- $\exists$ by tightness from particle system using $M1$-topology for Λ

Further prospects

- Application to neurosciences [Carrillo et al., D. et al.]
  - regard $-X$ as the firing potential of a neuron

    $\rightsquigarrow$ τ is the spiking time of the neuron

  - $\alpha \rightsquigarrow$ excitation parameter $\Rightarrow$ neurons are more likely to fire when one of them has spiked

- Application to finance [Hambly et al., N. S.]
  - regard X as the wealth of a company

    $\rightsquigarrow$ τ is the default time of the company

  - $\alpha \rightsquigarrow$ intensity of the default

- More general types of noise
  - how do the fluctuations impact the singularity?

    $\rightsquigarrow$ may have connection with mean field rough equations [Cass Lyons, Deuschel et al., Bailleul et al.]

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  - $\alpha \rightsquigarrow$ intensity of the default
- More general types of noise
  - may put a common noise

    $\rightsquigarrow$ get an SPDE with a free boundary [Hambly Ledger Sojmark]

Main results

- Assume: $u(0,\cdot)$ is bounded and changes monotonicity finitely often on compacts
  - take physical solution (X,Λ)
- Then: for any $t>0$, $p(t-,\cdot)$ has two same properties as $u(0,\cdot)$

(i) If $\limsup_{x\downarrow 0} x^{-1}p(t-,x)<\infty$, then $\Lambda\in C^1([t,t+\epsilon))$ for some $\epsilon>0$

(ii) If $\limsup_{x\downarrow 0} x^{-1}p(t-,x)=\infty$ but $\lim_{x\downarrow 0} p(-t,x)<\frac{1}{\alpha}$, then Λ is 1/2-Hölder continuous on $[t,t+\epsilon)$ for some $\epsilon>0$

(iii) If $\lim_{x\downarrow 0} p(t-,x)\geq\frac{1}{\alpha}$, then may jump

In all cases, $\exists\epsilon>0:\Lambda\in C^1((t,t+\epsilon))$ and $p(s,\cdot)$, $s\in(t,t+\epsilon)$, solves

$$\partial_t p=\frac{1}{2}\partial_{xx}p+\dot{\Lambda}_t\partial_x p,\quad p(\cdot,0)=0\ \text{ on }\ (t,t+\epsilon),\qquad \dot{\Lambda}_s=\frac{\alpha}{2}\partial_x p(s,0)$$

- Moreover: uniqueness

Part II. Elements of proof

C.d.f. estimates

- Step 0 : There is always a density! Smooth away from the front
  - shift of Brownian motion up to τ

C.d.f. estimates

- Step 0 : There is always a density! Smooth away from the front
- Step 1 : No possible jump of Λ in $(t, t+\epsilon)$, i.e.

  ◦ $\underbrace{\mathbb{P}(\tau \geq s,\ X_{s-} \leq x)}_{\text{remaining mass } \leq x} \leq \frac{\beta(z)}{\alpha} x \qquad x \leq \delta,\ s \in [t+z, t+\epsilon],\ \beta(z) < 1$

(i) if $\lim_{\eta \downarrow 0} \sup_{x \in (0,\eta)} p(t-, x) < 1/\alpha$

C.d.f. estimates

- **[Step 0]**: There is always a density! Smooth away from the front
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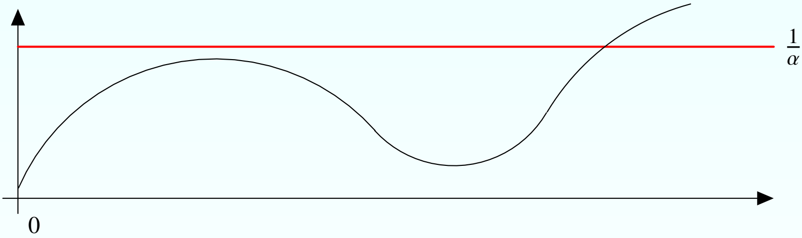

C.d.f. estimates

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(i) if $\lim_{\eta \downarrow 0} \sup_{x \in (0,\eta)} p(t-, x) < 1/\alpha$

(ii) if $p(t-, \cdot)$ locally monotone in right neighborhood of any $x > 0$

C.d.f. estimates

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- Step 1: No possible jump of Λ in $(t, t+\epsilon)$, i.e.

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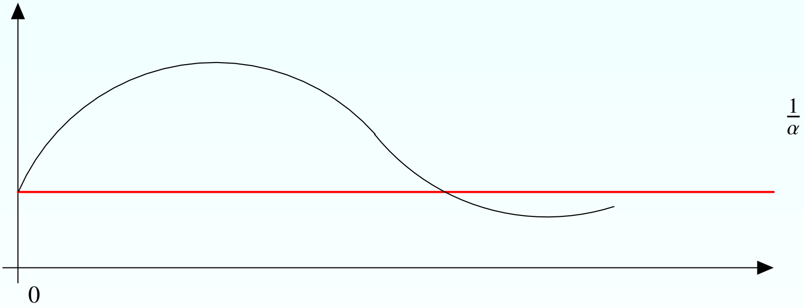

C.d.f. estimates

- Step 0 : There is always a density! Smooth away from the front
- Step 1 : No possible jump of Λ in $(t, t+\epsilon)$, i.e.

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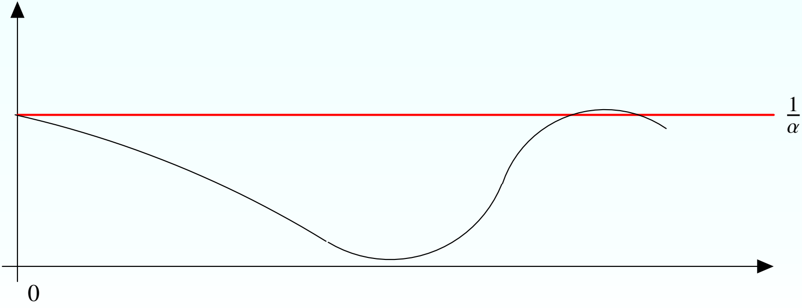
after jump



C.d.f. estimates

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  (i) if $\lim_{\eta \downarrow 0} \sup_{x \in (0,\eta)} p(t-, x) < 1/\alpha$

  (ii) if $p(t-, \cdot)$ locally monotone in right neighborhood of any $x > 0$

- Proof

$$
\begin{aligned}
&\mathbb{P}(\tau \geq s,\ X_{s-} \leq x) \\
&\leq \int \mathbb{P}(\Lambda_{s-} - \Lambda_{t-} \leq y + B_s - B_t \leq x + \Lambda_{s-} - \Lambda_{t-}) p(t-, y) dy \\
&= \int [F(x + \Lambda_{s-} - \Lambda_{t-} - z) - F(\Lambda_{s-} - \Lambda_{t-} - z)] g(s-t, z) dz
\end{aligned}
$$

C.d.f. estimates

- Step 0 : There is always a density! Smooth away from the front
- Step 1 : No possible jump of Λ in $(t, t+\epsilon)$, i.e.
  - $\underbrace{\mathbb{P}(\tau \geq s,\, X_{s-} \leq x)} \leq \frac{\beta(z)}{\alpha} x \qquad x \leq \delta,\, s \in [t+z, t+\epsilon],\, \beta(z) < 1$

(i) if $\lim_{\eta \downarrow 0} \sup_{x \in (0,\eta)} p(t-, x) < 1/\alpha$

(ii) if $p(t-, \cdot)$ locally monotone in right neighborhood of any $x > 0$

- Proof

$$\mathbb{P}(\tau \geq s,\, X_{s-} \leq x)$$

$$= \int \underbrace{[F(x + \Lambda_{s-} - \Lambda_{t-} - z) - F(\Lambda_{s-} - \Lambda_{t-} - z)]}_{\text{only local behavior counts}} g(s-t, z) dz$$

  - F c.d.f. of $p(t-, \cdot)$

    $\rightsquigarrow$ case (i) F is locally $< 1/\alpha$ Lipschitz

    $\rightsquigarrow$ case (ii) F becomes locally $< 1/\alpha$ Lipschitz after 0

Regularity estimates

- Step 1: If $\mathbb{P}(\tau \geq s,\, X_{s-} \leq x) \leq \dfrac{\beta}{\alpha} x$, $x \leq \delta$, $s \in [t, t+\epsilon]$, $\beta < 1$

  $\Rightarrow$ Λ is 1/2-Hölder on $[t, t+\epsilon]$

Regularity estimates

- Step 1: If $\mathbb{P}(\tau \geq s,\, X_{s-} \leq x) \leq \frac{\beta}{\alpha} x,\, x \leq \delta,\, s \in [t, t+\epsilon], \beta < 1$

  $\Rightarrow$ Λ is 1/2-Hölder on $[t, t+\epsilon]$

- Proof $\Lambda_s - \Lambda_t$ advance of front between t and s

$$\begin{aligned}
\Lambda_s - \Lambda_t &\leq \alpha \int \mathbb{P}\Big(y + \inf_{r \in [t,s]} \{B_s - B_t\} \leq \Lambda_s - \Lambda_t\Big) p(t,y)dy \\
&\leq \alpha \int_0^{\Lambda_s - \Lambda_t} \cdots + \alpha \int_{\Lambda_s - \Lambda_t}^{\infty} \cdots \\
&\leq \beta(\Lambda_s - \Lambda_t) + 2 \int_0^{\infty} \Phi\Big(\frac{y}{\sqrt{s-t}}\Big) p(t, y + \Lambda_s - \Lambda_t) dy \\
&\leq \beta(\Lambda_s - \Lambda_t) + C\sqrt{s-t}
\end{aligned}$$

$\rightsquigarrow$ Φ Gaussian survival function

Regularity estimates

- Step 1: If $\mathbb{P}(\tau \geq s,\, X_{s-} \leq x) \leq \frac{\beta}{\alpha} x$, $x \leq \delta$, $s \in [t, t+\epsilon]$, $\beta < 1$

  $\Rightarrow \Lambda$ is 1/2-Hölder on $[t, t+\epsilon]$

- Step 2: If Λ is 1/2-Hölder on $[t, t+\epsilon]$

  $\Rightarrow p(s,x) \leq Cx^{\chi}$ for $s \in [t+\epsilon/2, t+\epsilon]$ and $x \leq \delta$

- Proof p satisfies Fokker-Planck $\rightsquigarrow$ Feynman-Kac

$$p(s,x) = \mathbb{E}\Big[p(s-\rho, Y_\rho) \Big| Y_0 = x\Big]$$

  $\rightsquigarrow$ where $dY_r = \alpha \dot{\Lambda}_{s-r} dr + dB_r$

  $\rightsquigarrow$ $\rho = \inf\{r \geq 0 : Y_r \notin (0,\delta)\} \wedge \delta^2$, $\delta \ll 1$, $x \leq \delta/2$

  ◦ regularity of p at the boundary $\leftrightsquigarrow \mathbb{P}\{Y_\rho = 0\}$

$$p(s,x) \leq \mathbb{P}(\rho \geq \delta^2) \sup_{r \in [0,\delta^2], y \in [0,\delta]} p(s-r, y)$$

  ◦ probability to hit the boundary $\rightsquigarrow$ competition between B and Λ

  $\rightsquigarrow$ but Λ 1/2 Hölder $\Rightarrow B$ wins with >0 probability

Regularity estimates

- Step 1: If $\mathbb{P}(\tau \geq s,\, X_{s-} \leq x) \leq \frac{\beta}{\alpha} x$, $x \leq \delta$, $s \in [t, t+\epsilon]$, $\beta < 1$

  $\Rightarrow \Lambda$ is $1/2$-Hölder on $[t, t+\epsilon]$

- Step 2: If Λ is $1/2$-Hölder on $[t, t+\epsilon]$

  $\Rightarrow p(s,x) \leq C x^{\chi}$ for $s \in [t+\epsilon/2, t+\epsilon]$ and $x \leq \delta$

- Step 4: If Λ is $1/2$-Hölder on $[t, t+\epsilon]$

  $\Rightarrow p(s,x) \leq Cx$ for $s \in [t+\epsilon/2, t+\epsilon]$ and $x \leq \delta$ and p is smooth

- Proof:

  ◦ pass from Hölder decay from Lipschitz with barrier lemma (comparison of solutions)

  ◦ p Lipschitz at the boundary $\Rightarrow \Lambda$ Lipschitz

  ◦ X is a standard drifted Brownian motion

Propagation of monotonicity

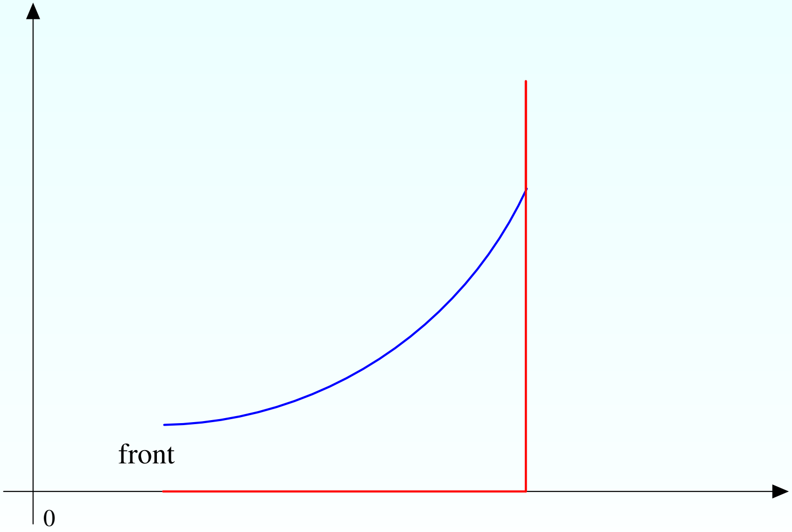
- monotonicity propagates if $\sharp$(sign changes $\partial_x u(t,\cdot)$) are controlled

  ◦ $u(t,x)$ is analytic in $x > \Lambda_t \Rightarrow$ zeros of $\partial_x u(t,x)$ are isolated in x away from the front

  $\rightsquigarrow$ propagation of the zeros in time!

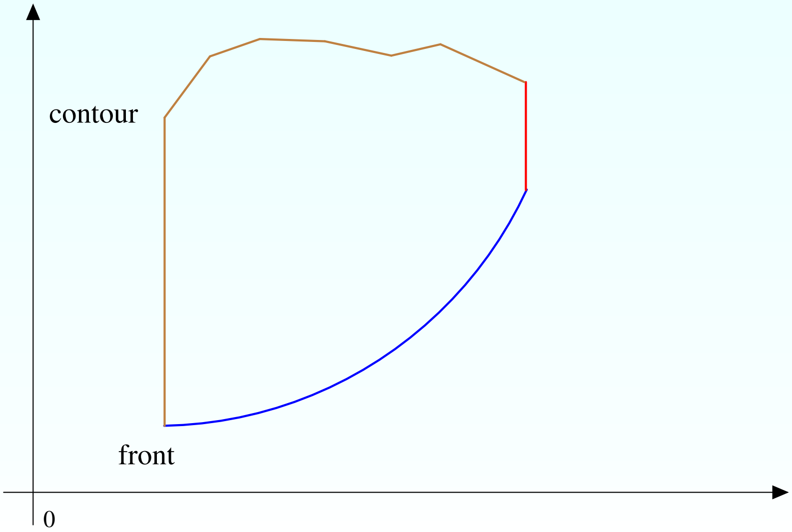
Propagation of monotonicity

- monotonicity propagates if $\sharp$(sign changes $\partial_x u(t,\cdot)$) are controlled
  - take an interval and control sign changes at right boundary



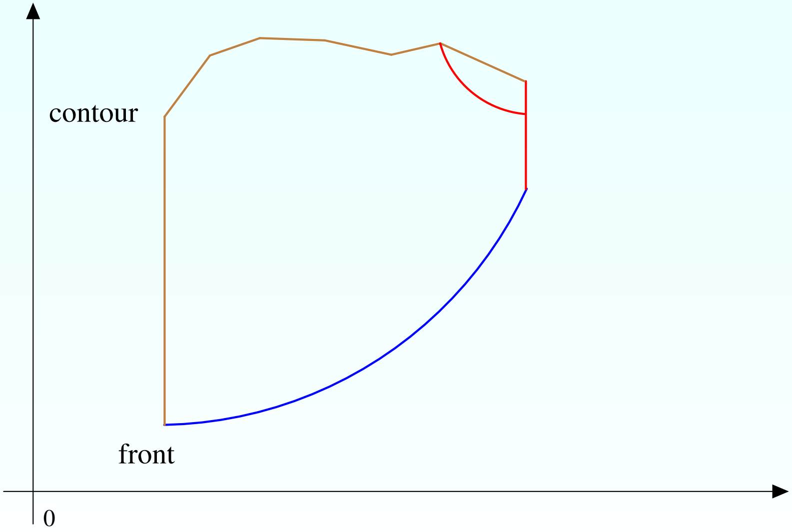
Propagation of monotonicity

- monotonicity propagates if $\sharp$(sign changes $\partial_x u(t,\cdot)$) are controlled
  - take a contour with a finite number of zeros ($\approx \sharp$ sign changes)



Propagation of monotonicity

- monotonicity propagates if $\sharp$(sign changes $\partial_x u(t,\cdot)$) are controlled
    - from starting point, may draw a minimal curve of zeros



Propagation of monotonicity

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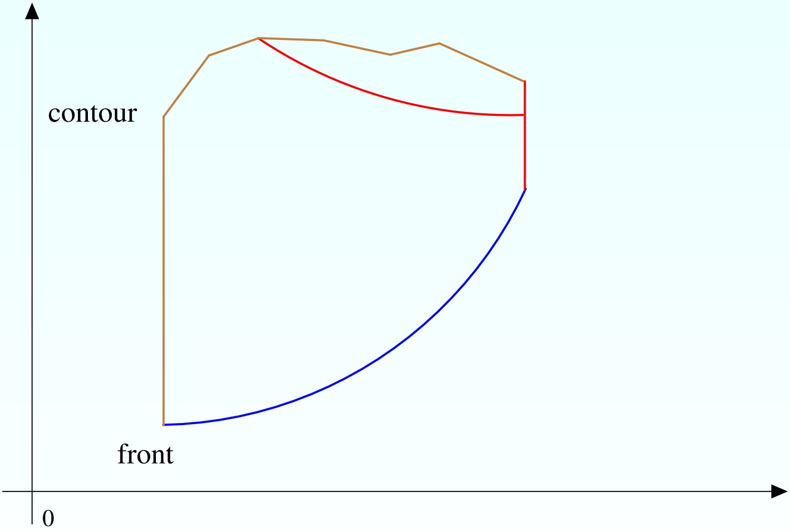
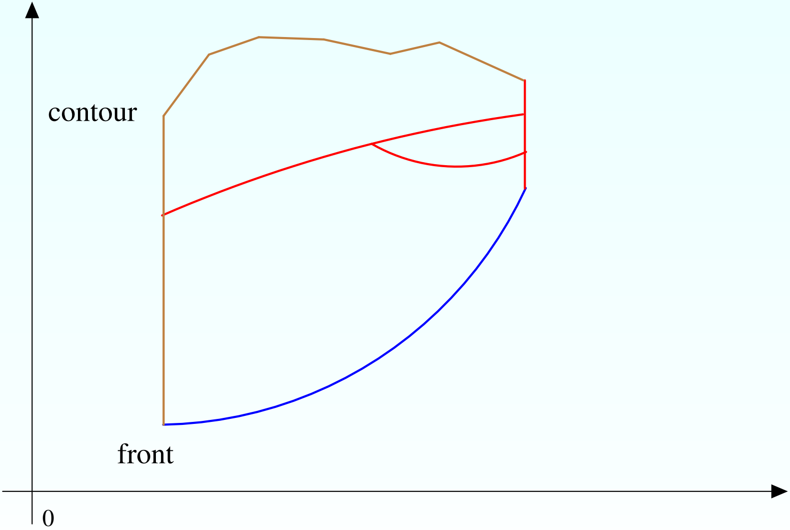


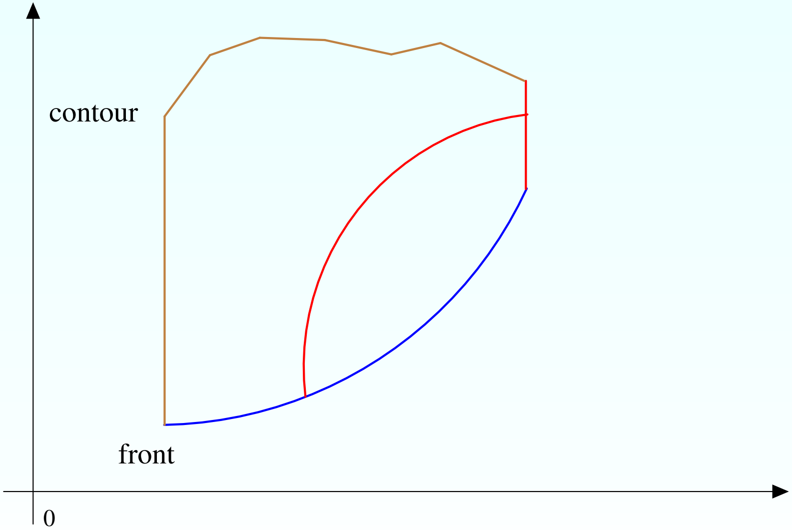
Propagation of monotonicity

- monotonicity propagates if $\sharp$(sign changes $\partial_x u(t, \cdot)$) are controlled
  - by max principle, curves hitting the contour cannot meet



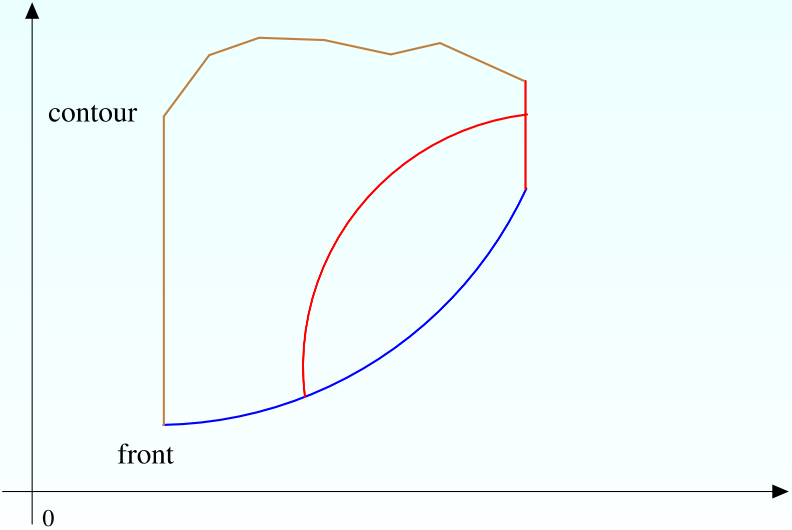
Propagation of monotonicity

- monotonicity propagates if $\sharp$(sign changes $\partial_x u(t, \cdot)$) are controlled
  - ◦ problem when zero curve touches the front



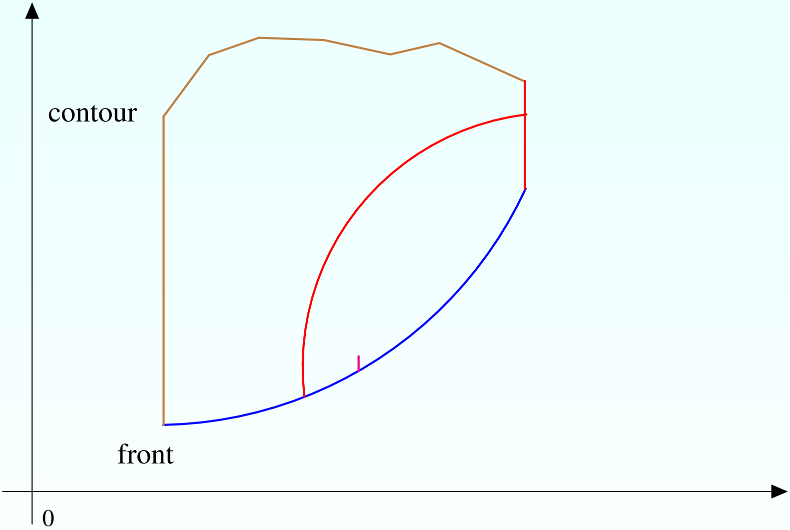
Propagation of monotonicity

- monotonicity propagates if ♯(sign changes $\partial_x u(t, \cdot)$) are controlled
  - sign locally preserved under the curve: prove it ≥ 0



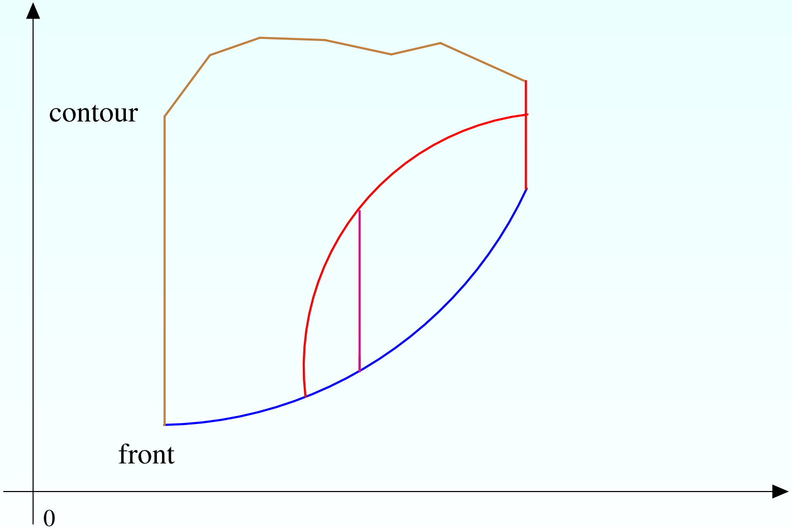
Propagation of monotonicity

- monotonicity propagates if $\sharp$(sign changes $\partial_x u(t,\cdot)$) are controlled
  - after hitting point of front by curve: u smooth, $\partial_x u < 0$ locally



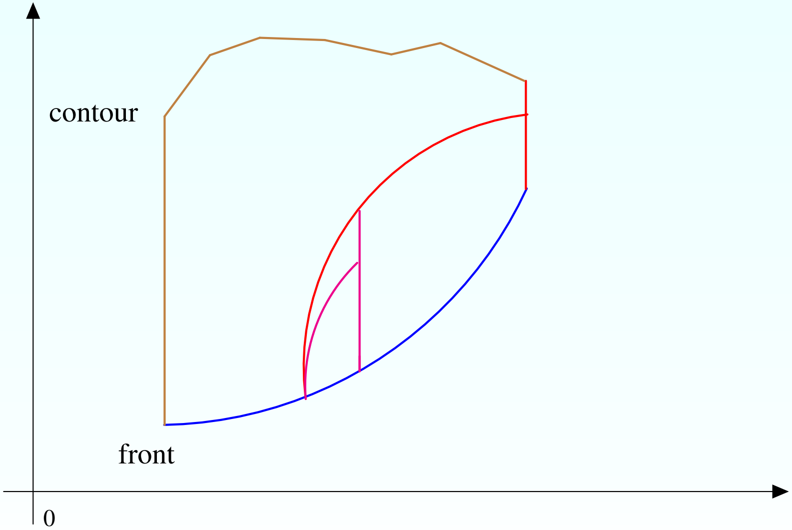
Propagation of monotonicity

- monotonicity propagates if $\sharp$(sign changes $\partial_x u(t,\cdot)$) are controlled
  - claim: $\partial_x u < 0$ globally: argue by contradiction



Propagation of monotonicity

- monotonicity propagates if ♯(sign changes $\partial_x u(t,\cdot)$) are controlled
  - a new $\partial_x u = 0$ would contradict maximum principle



Uniqueness

- Get a solution that is smooth except at some isolated times
  - enters smooth regime after any singularity
  - uniqueness by stability arguments
- Take two solutions (X, Λ) and (X', Λ')
  - they satisfy main estimates! prove local uniqueness after 0

using the sole assumptions on $u(0, \cdot)$

$$
\begin{aligned}
&|\Lambda - \Lambda'|_{[0,t]} \\
&\quad \leq \alpha \Big| \mathbb{P}\Big(\inf_{s\in[0,t]} (X_{0-} + B_s - \Lambda_s) \leq 0 \Big) - \mathbb{P}\Big(\inf_{s\in[0,t]} (X_{0-} + B_s - \Lambda'_s) \leq 0 \Big) \Big| \\
&\quad \leq \alpha \mathbb{P}\Big(0 \leq \inf_{s\in[0,t]} (X_{0-} + B_s - \Lambda'_s) \leq |\Lambda - \Lambda'|_{[0,t]} \Big) \\
&\qquad + \alpha \mathbb{P}\Big(0 \leq \inf_{s\in[0,t]} (X_{0-} + B_s - \Lambda_s) \leq |\Lambda - \Lambda'|_{[0,t]} \Big)
\end{aligned}
$$

Uniqueness

- Get a solution that is smooth except at some isolated times
  - enters smooth regime after any singularity
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- Take two solutions (X, Λ) and (X', Λ')
  - they satisfy main estimates! prove local uniqueness after 0 using the sole assumptions on $u(0, \cdot)$

$$|\Lambda - \Lambda'|_{[0,t]}$$

$$\leq \alpha \Big| \mathbb{P}\Big(\inf_{s \in [0,t]} (X_{0-} + B_s - \Lambda_s) \leq 0 \Big) - \mathbb{P}\Big(\inf_{s \in [0,t]} (X_{0-} + B_s - \Lambda'_s) \leq 0 \Big) \Big|$$

$$\leq \alpha \mathbb{P}\Big(0 \leq \inf_{s \in [0,t]} (X_{0-} + B_s - \Lambda'_s) \leq |\Lambda - \Lambda'|_{[0,t]} \Big)$$

$$+ \alpha \mathbb{P}\Big(0 \leq \inf_{s \in [0,t]} (X_{0-} + B_s - \Lambda_s) \leq |\Lambda - \Lambda'|_{[0,t]} \Big)$$

$$\leq |\Lambda - \Lambda'|_{[0,t]} - \Psi(|\Lambda - \Lambda'|_{[0,t]})$$

  - elaborate on (●) $\rightsquigarrow$ where Ψ is strictly positive on $(0, +\infty)$ (true for a small piece of time only)