Kolmogorov Equation on Wasserstein space and McKean Vlasov processes

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From joint works with N. Frikha
LPSM

Workshop Non linear processes
Saint-Etienne
2-5 juillet 2019

McKean-Vlasov SDE

On $[0, T]$, $T > 0$,

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s)ds + \int_0^t \sigma(s, X_s, \mu_s)dB_s, \quad X_0 \in L^2,$$

B is a B.M.

↳ Dynamic feels current state in physical space as well as its statistical distribution

↳ Sometimes called "distribution dependent" / "non-linear" / "mean-field" - SDE

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- Law given as the solution of non-linear Fokker-Planck equation (distributional sense)

$$\partial_t \mu_t + \mathrm{div}(\mu_t b(t, \cdot, \mu_t)) + [1/2] D^2 \big(\mu_t \sigma\sigma^*(t, \cdot, \mu_t)\big) = 0, \quad \mu_0 = [X_0]$$

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- Gives the asymptotic ($N \to +\infty$) dynamic of one particle interacting in mean field

$$\begin{cases} X_t^i = X_0 + \int_0^t b(s, X_s^i, \mu_s^N)ds + \int_0^t \sigma(s, X_s^i, \mu_s^N)dB_s^i, \quad i = 1, \dots, N, \\ \mu_s^N = N^{-1} \sum_{i=1}^N \delta_{X_s^i}, \quad X_0^i \sim X_0 \end{cases}$$

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↳ Popular for (*e.g.*) applications/connection in/with MFG, ANN

A MKV- SDE - well posedness

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- Cauchy-Lipschitz theory :

  ↳ Sensitive : two unknowns (position/distribution)

- Need to suitable choice of space and metric :

  ↳ space is $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

  ↳ metric is $(|\cdot - \circ|, \mathcal{W}_2(\cdot, \circ))$

$$\begin{aligned}
\mathcal{P}_p(\mathbb{R}^d) &= \left\{\mu \in \mathcal{P}(\mathbb{R}^d),\ s.t. \int |\cdot|^p d\mu < \infty\right\}, \quad p \geqslant 1 \\
\mathcal{W}_p(\mu, \mu') &= \inf_{\pi,\text{ coupling of } \mu,\mu'} \left\{\int |x - y|^p d\pi(x, y)\right\} \quad p \geqslant 1 \\
&\leqslant \mathbb{E}[|X - X'|^p],\ X \sim \mu,\ X' \sim \mu'
\end{aligned}$$

- W.P. ($\exists!$ path and law) for Lipschitz coefficients on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

MKV-SDE : ill vs well - posedness

- Beyond Cauchy-Lipschitz theory $\rightsquigarrow$ hard !

  ↳ Analogy with classical setting (Stroock&Varadhan M.P.'s theory / Zvonkin's theory)

(*i*) C.E. 1 (Sheutzow)

↳ $X_t = X_0 + \int_0^t \mathbb{E}[b(X_s)]ds$, b locally Lipschitz (even bounded !) $\rightsquigarrow$ may be ill-posed

↳ "relies on summation of local-Lipschitz constant over $\mathrm{supp}\{\mu\}$"

(*ii*) E. (Shiga and Tanaka) $\rightsquigarrow$ extensions (Jourdain - Mishura&Veretennikov - Lacker - Röckner & Zhang)

↳ $X_t = X_0 + \int_0^t \mathbb{E}[b(X_s)]ds + B_t$, b bounded locally Lipschitz (even only bounded !) $\rightsquigarrow \exists!$

↳ $\rightsquigarrow$ (drift continuous +bounded & Lip TV +diffusion linear & > 0)

↳ Noise may help ?

(*iii*) C.E. 2 (Delarue)

↳ $X_t = X_0 + \int_0^t b(\mathbb{E}[X_s])ds + B_t$, b bounded (even Hölder !)

$\rightsquigarrow$ may have several solutions !

↳ relies on ill posedness of $\dot{x}_t = b(x_t)$

- Finite dimensional noise to smooth infinite dimensional variable $\rightsquigarrow$ tricky smoothing properties $\rightsquigarrow$ need to investigate associated PDE

- $W_1(\mu,\nu) = \sup\left\{\left|\int h d\mu - \int h d\nu\right|, \quad \|h\|_{\mathrm{Lip}} \leqslant 1\right\}$(Kantorovitch)
- $\mathrm{TV}(\mu,\nu) = \frac{1}{2}\sup\left\{\left|\int h d\mu - \int h d\nu\right|, \quad \|h\|_{\infty} \leqslant 1\right\} = \frac{1}{2}\|p^{\mu} - p^{\nu}\|_{L_1}$

↳ $\mathcal{K}$ compact subset of $\mathbb{R}^d$,

$$\mu,\nu \in \mathcal{P}_2(\mathcal{K}), \quad W_1(\mu,\nu) \leqslant \mathrm{diam}(\mathcal{K})\mathrm{TV}(\mu,\nu)$$

↳ For coefficients $c(\mu) = \int \varphi d\mu \rightsquigarrow$ no need of regularity on φ to be Lipschitz in TV.

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MKV & Mean Field SDE- Chaos propagation

Recall that on $[0, T]$, $T > 0$,

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B is a B.M. gives asymptotic ($N \to +\infty$) dynamic of one particle interacting in mean field

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- Coefficients Lipschitz on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ $\rightsquigarrow$ propagation of chaos at the level of path (Sznitman) :
$$\bar{X}^i = \text{MKV-SDE}(X_0^i, B^i), \quad \mathbb{E}[\sup_{t \leqslant T} |\bar{X}_t^i - X_t^i|^2] \leqslant N^{-1}$$
- Coefficients Lipschitz on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ $\rightsquigarrow$ propagation of chaos at the level of semigroup (Carmona-Delarue) :
$$\forall \text{ smooth } \phi \quad \mathbb{E}[\sup_{t \leqslant T} |\phi(\mu_t^N) - \phi(\mu_t)|] \leqslant N^{-1/(d \vee 2)} \quad (\text{up to log for } d = 2)$$
- Coefficients continuous, bounded on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ + Lipschitz in TV + $\sigma > 0$ & "linear"
↳ propagation of chaos in TV between k-uplet - no rate - (Lacker)

$\rightsquigarrow$ Noise restores propagation of chaos ? $\rightsquigarrow$ Need to investigate smoothing properties

McKean-Vlasov SDE - associated PDE

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- Search for generator $\mathscr{L}$

  ↳ Solution (X, μ) is Markov on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

  ↳ Exists measurable map $u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ s.t.

$$\forall \varphi \text{ “smooth enough” } \forall t \in [0, T], \quad \mathbb{E}[\varphi(X_T, \mu_T) | (X_t, \mu_t)] = u(t, X_t, \mu_t).$$

- Dynamic of u ?
- generator should be PDE operator on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

  ↳ Derivative of map along flow of measure ?

  ↳ Itô's formula on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$?

Differentiability of functions of measure

Let $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$

(i) Work first on space of signed measure : flat or linear functional derivative $\frac{\delta}{\delta m} h$

↳ $\exists$ continuous function $[\delta h/\delta m] : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ s.t.

$$\lim_{\varepsilon \downarrow 0} \frac{h\big((1-\varepsilon)m + \varepsilon m'\big) - h(m)}{\varepsilon} = \int \frac{\delta h}{\delta m}(m)(y) d(m' - m)(y)$$

↳ defined up to additive constant $\rightsquigarrow$ choose $\int [\delta h/\delta m](m)(y) dm)(y) = 0$

(ii) Work with lift $\tilde{h} : L_2 \to \mathbb{R}$ of h : Lions' derivative $\partial_\mu h$

↳ Hilbert structure of $L_2 \rightsquigarrow$ Fréchet derivative $D\tilde{h}(X) =: \partial_\mu h(\mu)(X)$, law$(X) = \mu$

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$$\phi \in L_2(\mu), \quad \lim_{\varepsilon \downarrow 0} \frac{h\big(\mu \circ (\mathrm{Id} + \varepsilon \phi)\big) - h(\mu)}{\varepsilon} = \int \partial_\mu h(\mu)(y) \cdot \phi(y) d\mu(y)$$

- Example : φ smooth. $h : \mu \mapsto \int \varphi d\mu \rightsquigarrow \frac{\delta h}{\delta m} = \varphi, \quad \partial_\mu h = \nabla \varphi$

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↳ $h\big((1-\varepsilon)m + \varepsilon m'\big) - h(m) = \int \varphi(z) d[(1-\varepsilon)m + \varepsilon m' - m](z) = \varepsilon \int \varphi d(m' - m)$

↳ $Y \in L_2, \quad \tilde{h}(X + Y) - \tilde{h}(X) = \mathbb{E}[\varphi(X + Y) - \varphi(X)] = \mathbb{E}[\nabla \varphi(X) \cdot Y] + o(\|Y\|_{L_2})$

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- $[\delta h / \delta m]$ is bounded (sup-norm) $\rightsquigarrow$ h Lipschitz in total variation distance
- $\partial_\mu h$ is bounded (sup or L_2) $\rightsquigarrow$ h Lipschitz for $\mathcal{W}_1$ or $\mathcal{W}_2$

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Lion's derivative stronger notion than flat derivative : give regularity w.r.t. "weaker topology"

Differentiability of functions of measure : links

- Under suitable (and reasonable) assumptions

$$\partial_\mu h(\mu)(y) = \partial_y \frac{\delta h}{\delta m}(\mu)(y)$$

↳ Lions' derivative is gradient

↳ Lions' derivative requires more regularity

- Higher order of differentiations :

↳ Partial second order L-derivative :

$$\partial_y \partial_\mu h(\mu)(y) = \partial_y^2 \frac{\delta h}{\delta m}(\mu)(y)$$

↳ Full second order L-derivative :

$$\partial_\mu^2 h(\mu)(y) = \partial_y^2 \frac{\delta^2 h}{\delta m^2}(\mu)(y)$$

Need for chain rule ($\approx$ Itô's formula) on $\mathcal{P}_2$ (informal, for diff. proc.)

- Choose the correct notion of differentiation ?
  ↳ Need fo Itô's formula on $\mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow$ compute $[d/dt]h(\mu_t)$

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\begin{aligned}
& \frac{d}{dt} h(\mu_t) \\
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- Classical Itô's formula

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= \quad & \lim_{h \downarrow 0} \frac{1}{h} \int_{\mathbb{R}^d} \mathbb{E} \left[\frac{\delta h}{\delta m}(\mu_t)(X_{t+h}^{t,x}) - \frac{\delta h}{\delta m}(\mu_t)(x) \right] d\mu_t(x) \\
= \quad & \int_{\mathbb{R}^d} \mathbb{E}^{t,x} \left[\frac{d}{d\tau} \left(\frac{\delta h}{\delta m}(\mu_t)(X_\tau) \right) |_{\tau = t} \right] d\mu_t(x) \\
= \quad & \int_{\mathbb{R}^d} \mathbb{E}^{t,x} \left[b(t,x,\mu_t) \cdot \partial_y \frac{\delta h}{\delta m}(\mu_t)(X_t) + \frac{1}{2} \mathrm{Tr}[\sigma\sigma^*(t,x,\mu_t) \partial_y^2 \frac{\delta h}{\delta m}(\mu_t)(X_t)] \right] d\mu_t(x) \\
= \quad & \int_{\mathbb{R}^d} \left[b(t,x,\mu_t) \cdot \partial_\mu h(\mu_t)(x) + \frac{1}{2} \sigma\sigma^*(t,x,\mu_t) \partial_y \partial_\mu h(\mu_t)(x) \right] d\mu_t(x)
\end{aligned}
$$

- Definition of flat derivative ($\mu_t^\lambda = \lambda\mu_t + (1-\lambda)\mu_{t+h}$)
- Markov property $\rightsquigarrow$ $\mu_{t+h} = \mu_t \star \mu_{t,t+h}$
- Classical Itô's formula
- link between flat and Lions' derivative

McKean-Vlasov SDE - associated Kolmogorov PDE on $\mathcal{P}_2$

On $[0, T]$, $T > 0$,

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s)ds + \int_0^t \sigma(s, X_s, \mu_s)dB_s, \quad X_0 \in L^2,$$

B is a B.M.

- Deduce through Itô's formula that associated Kolmogorov equation writes

$$\partial_t u(t,\mu) + \int b(t,x,\mu)\cdot \partial_\mu u(t,\mu)(x) d\mu(x) + \frac{1}{2}\int \mathrm{Tr}[(\sigma\sigma^*)(t,x,\mu)\partial_x \partial_\mu u(t,\mu)(x)] d\mu(x) = 0$$

  ↳ Well-posedness for smooth coefficients (Buckdahn&Li&Peng - Chassagneux&Crisan&Delarue - Crisan&McMurray).

- Smoothing properties ? $\rightsquigarrow$ fundamental solution ? If yes, which regularity ?
- Search for a map $[0, t) \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p(\mu, s, t, z)$, $(t, z) \in [0, T] \times \mathbb{R}^d$ s.t.
  - (i) For every fixed $(t, z) \in [0, T] \times \mathbb{R}^d$, the map $[0, t) \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p(\mu, s, t, z)$ satisfies

  $$(\partial_s + \mathcal{L}_s)p(\mu, s, t, z) = 0 \quad \text{on } [0, t) \times \mathcal{P}_2(\mathbb{R}^d).$$

  - (ii) For any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$

  $$\lim_{s \uparrow t} p(\mu, s, t, z) = \delta_z(.) \star \mu$$

- If it exists, p should be density of the MKV process ! $p(\mu, s, t, \cdot) = d\mu_t^{s,\mu}$

McKean-Vlasov SDE - density ?

- Consider MKV SDE with $\sigma > 0$

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s)ds + \int_0^t \sigma(s, X_s, \mu_s)dB_s, \quad X_0 \in L^2$$

- Introduce decoupled flow $X^{t,x,\mu}$: SDE **frozen** along the transport of μ along the flow $(\mu_s^{t,\mu})_{t\leqslant s\leqslant T}$

$$X_t^{t,x,\mu} = x + \int_t^s b(r, X_r^{t,x,\mu}, \mu_r^{t,\mu})dr + \int_0^t \sigma(r, X_r^{t,x,\mu}, \mu_r^{t,\mu})dB_s, \quad x \in \mathbb{R}^d$$

↳ Classical SDE $\rightsquigarrow$ admits (transition) density $p(t, \mu; t, x, s, \cdot)$
↳ Density $p(t, \mu; t, x, s, \cdot)$ admits **parametrix expansion** :

$$p(t, \mu; t, x, s, \cdot) = g\left(\int_t^s (\sigma\sigma^*)(r, \tilde{x}, \mu_r^{t,\mu})dr, \cdot - x - \int_t^s b(r, \tilde{x}, \mu_r^{t,\mu})dr\right) + \text{Remainder}\big(t, \mu, (s-t)\big)$$

↳ g is Gaussian kernel $\rightsquigarrow$ recovering (usual) smoothing on physical space

- Weak ∃! MKV SDE admits density

$$p(\mu, t, s, \cdot) = \int p(t, \mu; t, x, s, \cdot)d\mu(x)(= d\mu_s^{t,x}!)$$

↳ **parametrix expansion** around Gaussian kernel !
↳ Search for regularity in μ : Problem is circular !

MKV SDE - density : regularity in μ variable - building block

Let $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ with continuous and bounded flat derivative, flow $(\mu_t^{0,\mu})_{0 \leqslant t \leqslant T}$ given by unique weak solution of

$$X_t = \xi + B_t, \quad \xi \sim \mu$$

- Lions's differentiability of $\mu \mapsto h(\mu_t^{0,\mu})$?
  ↳ Writes $h(\mu_t^{0,\mu}) = h(\mu \star g_t)$, g_t gaussian kernel $\mathcal{N}(0,t)$, take μ, μ' in $\mathcal{P}_2(\mathbb{R}^d)$:

$$\begin{aligned} h(\mu_t^{0,\mu}) - h(\mu_t^{0,\mu'}) &= h(\mu \star g_t) - h(\mu' \star g_t) \\ &= \int_0^1 \int \frac{\delta h}{\delta m}(\lambda \mu_t^{0,\mu} + (1-\lambda)\mu_t^{0,\mu'})(y) g_t(y-x) d(\mu - \mu')(x) dy d\lambda \end{aligned}$$

  ⇝ Flat derivative is $\frac{\delta}{\delta m} h(\mu_t^{0,\mu})(x) = \int \frac{\delta h}{\delta m}(\mu_t^{0,\mu})(y) g_t(y-x) dy$

- $(\mu, x) \mapsto [\delta h / \delta m](\mu)(x)$ jointly continuous and bounded
  ↳ Recovering spatial smoothing : $x \mapsto \frac{\delta}{\delta m} h(\mu_t^{0,\mu})(x)$ is smooth !

  ⇝ h Lions' differentiable (first and second order partial derivative) ⇝ sufficient for chain rule !

  ⇝ First and second derivatives blow up at rate resp. $t^{-1/2}$ and t^{-1} ! ⇝ second order possibly too coarse !

MKV SDE / smoothing of MKV semigroup - zoology

Let $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, flot $(\mu_t^{0,\mu})_{0 \leqslant t \leqslant T}$ given by unique weak solution of

$$dX_t = dB_t, \quad X_0 = \xi \sim \mu$$

- Regularization by smooth flow of probability measure : $\mu \mapsto h(\mu)$ "only" flat differentiable with bounded and continuous flat derivative :

  $\rightsquigarrow$ $\mu \mapsto h(\mu_t^{0,\mu})$ L-differentiable (first and partial 2^{nd} order !), blow up at resp. $t^{-1/2}$ and t^{-1}

  $\rightsquigarrow$ $\mu \mapsto h(\mu_t^{0,\mu})$ now Lipschitz w.r.t. d_1 where

$$d_\eta(\mu, \nu) = \inf_{\pi \text{ coupling}} \int \{|x-y|^\eta \wedge 1\} d\pi(x,y), \quad \eta \in (0,1]$$

  ↳ weakining of the topology : from TV to d and hence Wasserstein !

- Assume in addition $x \mapsto [\delta h/\delta m](\mu)(x)$ is η-Hölder continuous :

  ↳ h is now Lipschitz for d_η

  $\rightsquigarrow$ $\mu \mapsto h(\mu_t^{0,\mu})$ L-differentiable (first and partial 2^{nd} order !), blow up at resp. $t^{-(1+\eta)/2}$ and $t^{-1+\eta/2}$

  ↳ singularity is now integrable !

  ↳ $\mu \mapsto h(\mu_t^{0,\mu})$ now Lipschitz w.r.t. d_1 (and Wasserstein 1-2) weakining of the topology

- Previous (partial) results on smoothing by Banos - CdR - McMurray - Crisan&McMurray

MKV SDE - smoothness of density

- Consider MKV SDE with $\sigma > 0$, **coefficients admit twice bounded continuous flat derivative** + first and second flat derivatives Hölder continuous in space

$$dX_t = b(t, X_t, \mu_t)dt + \sigma(t, X_t, \mu_t)dB_t, \quad X_0 \in L_2$$
$$dX_t^{s,x,\mu} = b(t, X_t^{s,x,\mu}, \mu_t^{s,\mu})dt + \sigma(t, X_t^{s,x,\mu}, \mu_t^{s,\mu})dB_t, \quad X_0^{x,\mu} = x \in \mathbb{R}^d$$

$$p(t,\mu;t,x,s,\cdot) = g\left(\int_t^s (\sigma\sigma^*)(r,\tilde{x},\mu_r^{t,\mu})dr, \cdot - x - \int_t^s b(r,\tilde{x},\mu_r^{t,\mu})dr\right) + \mathrm{R}(t,\mu,(s-t))$$

- MKV SDE admits density $p(\mu,t,s,\cdot) = \int p(t,\mu;t,x,s,\cdot)d\mu(x)(= d\mu_s^{t,x}!)$
- Handle circular problem through Picard procedure :
  ↳ $\left\{\left((\mu_s^{t,\mu})_{t\leqslant s\leqslant T}\right)^{\ell}\right\}_{\ell\geqslant 0}$ through Picard iteration on MKV-SDE initialized at $\nu \neq \mu$

  ↳ $(p_\ell)_{\ell\geqslant 0}$ corresponding decoupled flow $\rightsquigarrow$ L-differentiability at step $\ell+1$ $\rightsquigarrow$ diff. + smoothing for flow at step $\ell+1$...

  ↳ Uniform control + equicontinuity $\rightsquigarrow$ converging subsequence through compactness
- $p(t,\mu,s,x,y)$ and $p(t,\mu,s,y)$ smooth in all variable + Gaussian type bounds

  ↳ Blow up smoothed by $\eta/2$ for L-derivative

From smoothing to W.P. for non-degenerate MKV - SDE - use of $\exists$ of density

On $[0, T]$, $T > 0$,

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s) ds + \int_0^t \sigma(s, X_s, \mu_s) dB_s, \quad X_0 \in L^2,$$

B is a B.M.

- Correct framework seems to be coefficients with bounded and continuous flat derivative (possibly need of Hölder regularity in space and for flat derivative) $\rightsquigarrow$ no results in that direction
- Idea : use existence of density and **parametrix expansion** (order 1)
  ↳ Space $= \{\mathbf{P} \in \mathcal{C}([s, T], \mathcal{P}(\mathbb{R}^d), \mathbf{P}(s) = \mu, \mathbf{P}(t)$ with density $p, TV\}$
  ↳ Compute $[\delta/\delta m]p$ and show it is bounded
  ↳ Derive Lipschitz in TV $\rightsquigarrow$ fixed point procedure converges
  $\rightsquigarrow$ Need for Hölder continuity for $\sigma\sigma^*$ to handle **remainder of parametrix expansion** !
- **Result** Under these assumptions $\exists!$ weak sol of MKV-SDE
  ↳ Work for bounded drift Lipschitz in TV
  ↳ Gives strong $\exists!$ asa σ Lipschitz in space

Examples

On $[0, T]$, $T > 0$,

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s)ds + \int_0^t \sigma(s, X_s, \mu_s)dB_s, \quad X_0 \in L^2,$$

B is a B.M.

$\rightsquigarrow$ Weak W.P. for

- M order interaction $h(t, x, \mu) = \int \varphi(t, x, z_1, \dots, z_M) d\mu(z_1) \dots d\mu(z_M)$

  ↳ φ measurable and bounded + η-Hölder continuous

- Scalar interaction $h(t, x, \mu) = \int \varphi\Big(t, x, \int \varphi_1 d\mu, \dots, \int \varphi_M d\mu\Big)$

  ↳ φ measurable and bounded, $z \mapsto \varphi(t, x, \mathbf{z})$ Lipschitz + $x \mapsto \varphi(t, x, \mathbf{z})$ η-Hölder

  ↳ φ_i measurable and bounded + η-Hölder continuous

- Polynomial on Wasserstein space $h(t, x, \mu) = \prod_{i=1}^N \Big[\int \varphi_i(t, x, z) d\mu(z)\Big]$

  ↳ φ measurable and bounded + η-Hölder continuous

+Lipschitz in space $\rightsquigarrow$ Strong W.P.

MKV SDE - (back to) associated Kolmogorov PDE on $\mathcal{P}_2$

On $[0, T]$, $T > 0$,

$$\partial_t u(t,\mu) + \int b(t,x,\mu) \cdot \partial_\mu u(t,\mu)(x) d\mu(x) + \frac{1}{2} \int \mathrm{Tr}[(\sigma\sigma^*)(t,x,\mu)\partial_x \partial_\mu u(t,\mu)(x)] d\mu(x) = 0$$

Coefficients are bounded and η-Hölder in space and (twice) flat differentiable with bounded η-Hölder continuous derivative

- **Result** The backward Kolmogorov equation admits a unique fundamental solution $p(\mu, s, t, z)$ which writes

$$p(\mu, s, t, z) = \int p(t, \mu, s, x, t, z) d\mu(x)$$

  and (first and partial-second) L-derivatives admit Gaussian type bound with blow up at resp. $t^{-(1+\eta)/2}$ and $t^{-1+\eta/2}$

- **Result** Cauchy problem associated with backward Kolmogorov equation admits classical solution

$$u(t,\mu) = \phi(\mu_T^{t,\mu}) + \int_t^T \mathbf{f}(s, \mu_s^{t,\mu}) ds$$

  for any flat differentiable with bounded and Hölder continuous derivative source term and any bounded with continuous and bounded flat differentiable terminal condition

MKV SDE - (back to) associated PDE on $\mathbb{R}^d \times \mathcal{P}_2$

$$dX_t = b(t, X_t, \mu_t)dt + \sigma(t, X_t, \mu_t)dB_t, \quad X_0 \in L_2$$
$$dX_t^{s,x,\mu} = b(t, X_t^{s,x,\mu}, \mu_t^{s,\mu})dt + \sigma(t, X_t^{s,x,\mu}, \mu_t^{s,\mu})dB_t, \quad X_0^{x,\mu} = x \in \mathbb{R}^d$$

Coefficients are bounded and η-Hölder in space and (twice) flat differentiable with bounded η-Hölder continuous derivative

- Solution (X, μ) is Markov on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

  ↳ generator $\mathscr{L}$ acts on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow \mathscr{L} = L + \mathcal{L}$
- Search for dynamic of $u(t, X_t, \mu_t) = \mathbb{E}[\varphi(X_T, \mu_T)|(X_t, \mu_t)]$

  ↳ use decoupled flow $u(t, x, \mu) = \mathbb{E}[\varphi(X_T^{t,x,\mu}, \mu_T^{t,\mu})]$ + Markov + Itô to derive

$$(\partial_t + \mathscr{L})u(t, x, \mu) = 0, \quad u(T, \cdot, \cdot) = \varphi$$

- **Result** The Cauchy pb associated with $\mathscr{L}$ with data $(\mathbf{f}, \varphi)$ admits a unique classical solution

$$u(t, x, \mu) = \mathbb{E}[\varphi(X_T^{t,x,\mu}, \mu_T^{t,\mu}) + \int_t^T f(s, X_s^{t,x,\mu}, \mu_s^{t,\mu})ds]$$

  for bounded with bounded flat differentiable terminal condition and bounded and Hölder with bounded and Hölder flat differentiable source term

  ↳ works for unbounded t.c. and source provided suitable exponential growth in space and quadratic in μ

  ↳ works for source locally Hölder (space + flat derivative)

From PDE to Prop of chaos - path level

Recall that on $[0, T]$, $T > 0$,

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s) ds + \int_0^t \sigma(s, X_s, \mu_s) dB_s, \quad X_0 \in L^2,$$

B is a B.M. gives asymptotic ($N \to +\infty$) dynamic of one particle interacting in mean field

$$X_t^i = X_0^i + \int_0^t b(s, X_s^i, \mu_s^N) ds + \int_0^t \sigma(s, X_s^i, \mu_s^N) dB_s^i, \quad i = 1, \ldots, N, \ \mu_s^N = N^{-1} \sum_{i=1}^N \delta_{X_s^i}$$

- Restore propagation of chaos, idea
  ↳ Assume that σ is (in addition) Lipschitz in space, X_0 admit moment of order $q > 4$
  ↳Take u sol of $(\partial_t + \mathscr{L})u = b, \quad u_T = 0$
  ↳ Compute Zvonkin's transform of X_t^i and $\bar{X}_t^i = \text{MKV-SDE}(X_0^i, B^i)$
  ⇝ require to control second L-derivative ∂_μ^2 ! ⇝ need to work with Picard approximation !
  ↳ Compare path
- **Result.** Under assumptions for PDE on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ + Lip. diff. and moment one has

$$\mathbb{E}[\sup_{t \leqslant T} |X_t^i - \bar{X}_t^i|^2] \leqslant N^{-2/(d \vee 4)}, \quad \text{(up to log for } d = 4)$$

From PDE to Prop of chaos - density level

Recall that on $[0, T]$, $T > 0$,

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s) ds + \int_0^t \sigma(s, X_s, \mu_s) dB_s, \quad X_0 \in L^2,$$

B is a B.M. gives asymptotic ($N \to +\infty$) dynamic of one particle interacting in mean field

$$X_t^i = X_0^i + \int_0^t b(s, X_s^i, \mu_s^N) ds + \int_0^t \sigma(s, X_s^i, \mu_s^N) dB_s^i, \quad i = 1, \dots, N, \ \mu_s^N = N^{-1} \sum_{i=1}^N \delta_{X_s^i}$$

- pointwise propagation of chaos, idea

  ↳ $p^{1,N}$: marginal density of the first interacting particle

  ↳ Take p fundamental sol of Kolmogorov PDE and test against μ^N through Chain rule on $\mathcal{P}_2$

  ↳ Use fact that p is fundamental solution

  ⇝ require to control second L-derivative ∂_μ^2 ! ⇝ need to work with Picard approximation !

  ⇝ leads to $|(p^{1,N} - p)(\mu, 0, t, z)| \leqslant |\mathbb{E}[p(\mu_0^N, 0, t, z) - p(\mu, 0, t, z)]|$ + Remainder

  ↳ Conclusion thanks to regularity on p

- **Result.** Under assumptions for PDE on $\mathcal{P}_2(\mathbb{R}^d)$ one has

$$|(p^{1,N} - p)(\mu, 0, t, z)| \leqslant \frac{K}{N} \left\{ \frac{1}{t^{\frac{1-\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z - x)|x| d\mu(x) + \frac{1}{t^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z - x) d\mu(x) \right\}$$

From PDE to Prop of chaos - semigroup level

Recall that on $[0, T]$, $T > 0$,

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s)ds + \int_0^t \sigma(s, X_s, \mu_s)dB_s, \quad X_0 \in L^2,$$

B is a B.M. gives asymptotic ($N \to +\infty$) dynamic of one particle interacting in mean field

$$X_t^i = X_0^i + \int_0^t b(s, X_s^i, \mu_s^N)ds + \int_0^t \sigma(s, X_s^i, \mu_s^N)dB_s^i, \quad i = 1, \dots, N, \; \mu_s^N = N^{-1}\sum_{i=1}^N \delta_{X_s^i}$$

- propagation of chaos, idea
  ↳ Take U sol of Kolmogorv PDE and test against μ^N through Chain rule on $\mathcal{P}_2$
  ↳ Use fact that U solves Kolmogorov PDE on Wasserstein space
  ⇝ require to control second L-derivative ∂_μ^2 ! ⇝ need to work with Picard approximation !
  ⇝ leads to $|(U(t, \mu_t^N) - U(t, \mu)| \leqslant |\mathbb{E}[U(0, \mu_0^N) - U(0, \mu_0]| + \text{Remainder} \approx \frac{C}{N}$
  ↳ Conclusion thanks to regularity on U
- **Result.** Under assumptions for PDE on $\mathcal{P}_2(\mathbb{R}^d)$ one has, for all ϕ in $\mathscr{C}_{b,\alpha}^2$

$$\mathbb{E}[|\phi(\mu_T^N) - \phi(\mu_T)|] \leqslant CT^{\frac{-1+\alpha}{2}} W_1(\mu_0, \mu_0^N), \quad |\mathbb{E}[\phi(\mu_T^N)]| - \phi(\mu_T)| \leqslant CT^{-1+\frac{\alpha}{2}} \frac{C}{N}$$

↳ First order expansion through additional assumptions

From PDE to Prop of chaos - semigroup level

Recall that on $[0, T]$, $T > 0$,

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s)ds + \int_0^t \sigma(s, X_s, \mu_s)dB_s, \quad X_0 \in L^2,$$

B is a B.M. gives asymptotic ($N \to +\infty$) dynamic of one particle interacting in mean field

$$X_t^i = X_0^i + \int_0^t b(s, X_s^i, \mu_s^N)ds + \int_0^t \sigma(s, X_s^i, \mu_s^N)dB_s^i, \quad i = 1, \ldots, N, \ \mu_s^N = N^{-1}\sum_{i=1}^N \delta_{X_s^i}$$

- propagation of chaos, idea
  ↳ Take U sol of Kolmogorv PDE and test against μ^N through Chain rule on $\mathcal{P}_2$
  ↳ Use fact that U solves Kolmogorov PDE on Wasserstein space
  ⇝ require to control second L-derivative ∂_μ^2 ! ⇝ need to work with Picard approximation !
  ⇝ leads to $|(U(t, \mu_t^N) - U(t, \mu)| \leqslant |\mathbb{E}[U(0, \mu_0^N) - U(0, \mu_0]| + \text{Remainder} \approx \frac{C}{N}$
  ↳ Conclusion thanks to regularity on U
- **Result.** Under assumptions for PDE on $\mathcal{P}_2(\mathbb{R}^d)$ one has, for all ϕ in $\mathscr{C}_{b,\alpha}^2$

$$\mathbb{E}[|\phi(\mu_T^N) - \phi(\mu_T)|] \leqslant CT^{\frac{-1+\alpha}{2}} W_1(\mu_0, \mu_0^N), \quad |\mathbb{E}[\phi(\mu_T^N)]| - \phi(\mu_T)| \leqslant CT^{-1+\frac{\alpha}{2}}\frac{C}{N}$$

  ↳ First order expansion through additional assumptions
  ↳ Works for $\phi = \int \varphi d\mu$ with φ measurable and bounded only !

Thank you !