Random walk in a non-integrable random scenery time

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joint work with Marco Lenci & Françoise Pène

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Nonlinear Processes and their Applications

Outline

Anomalous diffusions

Anomalous diffusions are stochastic processes $X(t) \in \mathbb{R}^d$ that scale in time with exponent $\delta \neq 1/2$:

$$\mathbb{E}(|X(t)|^2) \sim t^{2\delta} \quad \text{for } t \to \infty, \quad \delta \neq 1/2$$

The behavior of superdiffusive processes ($\delta > 1/2$) characterizes many different natural systems and is mainly connected to motion in disorder media:

- light particle in an optical lattice;
- tracer in a turbolent flow;
- molecular diffusion in porous media.

Main features

- long ballistic “flights“
- short disorder motion

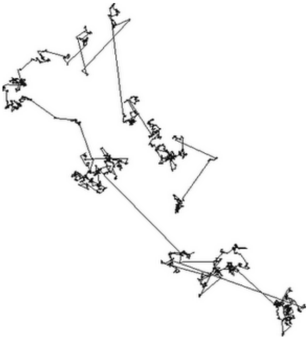

Figura: Typical Lévy flight

Models for anomalous diffusions

Schlesinger, Klafter['85]; Zaburdaev, Denisov, Klafter ['15]; Dybiec, Gudowska-Nowak, Barkai, Dubkov ['17]

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Random walk on $\mathbb{R}^d$ with jumps length given by a sequence of i.i.d. α-stable- r.v., with $\alpha \in (0, 2)$.

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Good behavior but **naive models**: the lengths of the jumps are independent $\Longrightarrow$ the medium is renewed after each jump.

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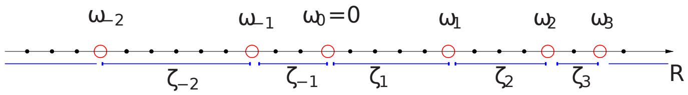
- Define the environment $\omega = \{\omega_k\}_{k\in\mathbb{Z}}$ as the renewal P.P. on $\mathbb{R}$

$$\omega_0 = 0\,, \quad \omega_k - \omega_{k-1} = \zeta_k$$ (Lévy) Random environment

with $(\zeta_k)_{k\in\mathbb{Z}\setminus\{0\}}$ i.i.d. positive r.v. :
$n^{-1/\alpha}(\zeta_1 + \ldots + \zeta_n) \xrightarrow{d} Z_1$ with Z_1 α-stable r.v., $\alpha \in (0,2)$.



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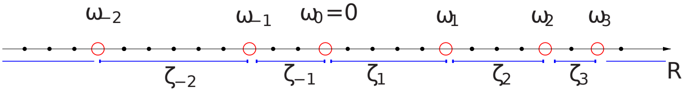

- Let $(\xi_j)_{j\in\mathbb{N}}$ i.i.d. integer r.v.'s with $\mathbb{E}(\xi_1) = 0$ and $\mathbb{E}(\xi_1^2) < \infty$:

$$S_0 = 0\,, \quad S_n = \sum_{j=1}^{n} \xi_j \qquad \text{underlying RW on } \mathbb{Z}$$

Discrete and continuous time processes

Discrete time process $Y = (Y_n)_{n \in \mathbb{N}}$ is the RW on ω coupled to S,

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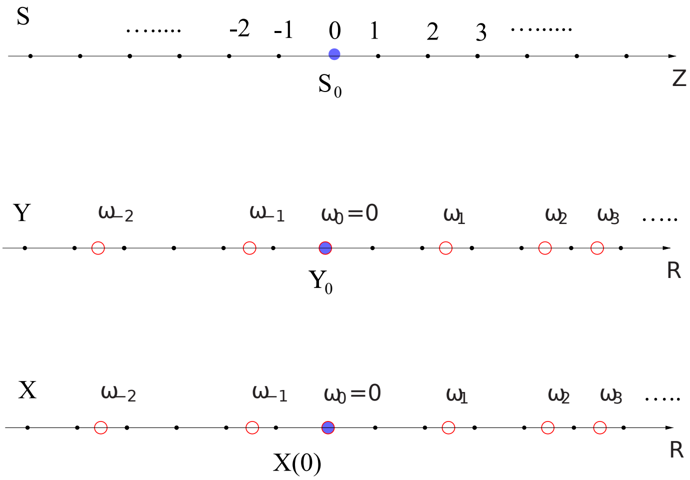
- Set $X_t := Y_n + \operatorname{sgn}(\xi_{n+1})(t - T_n)\,, \qquad \text{for } t \in [T_n, T_{n+1})$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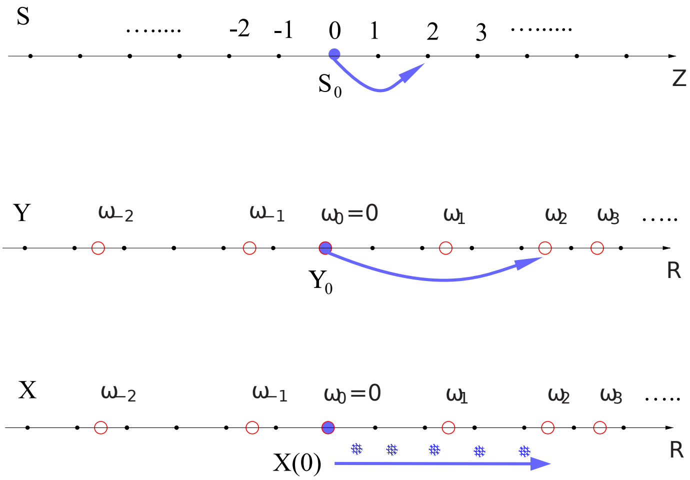
R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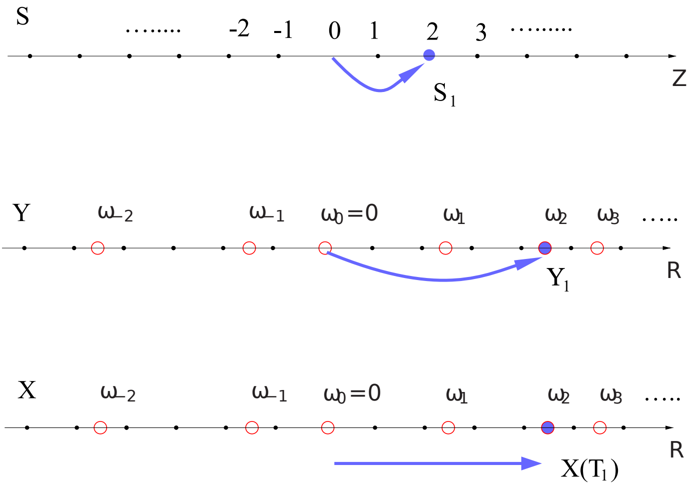
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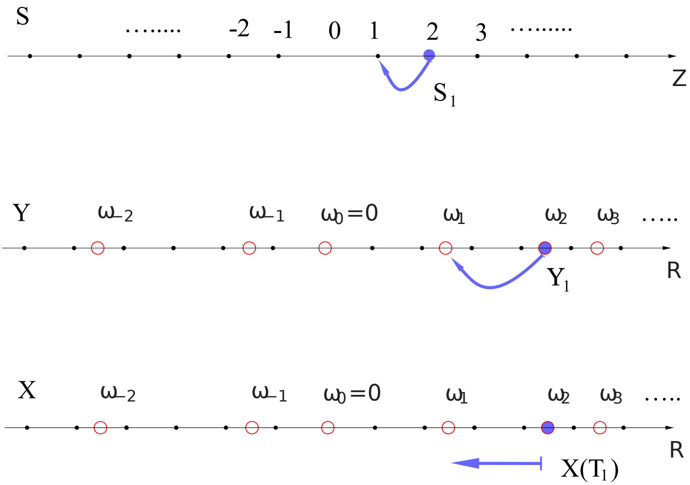
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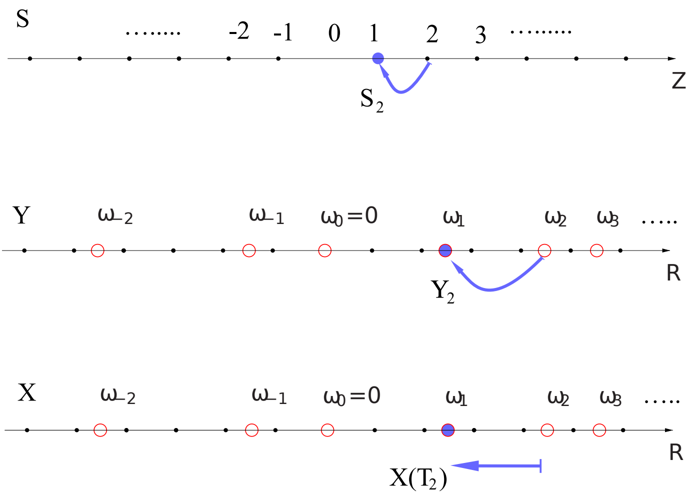
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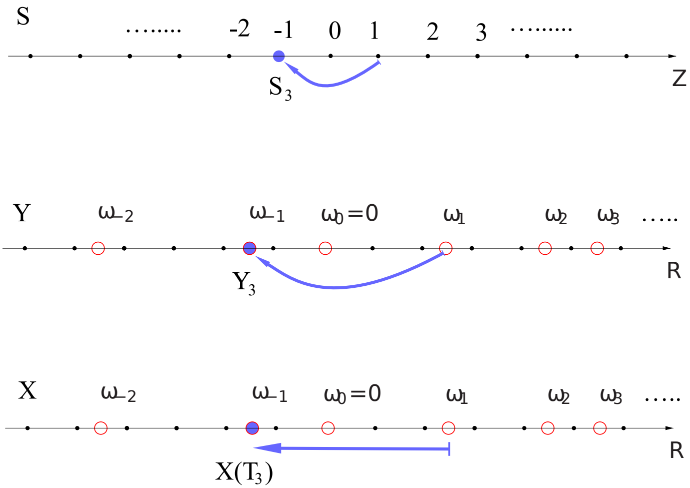
R
$X(T_2)$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We will consider:

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Goal: Scaling limit of $(Y_n)_{n\in\mathbb{N}}$ and $(X_t)_{t\in\mathbb{R}^+}$.

Previous (annealed) results

- Annealed second moment $\mathbb{E}(X_t^2)$
  (Barkai, Fleurer, Klafter ['00], Burioni, Caniparoli, Vezzani ['10])

$$\mathbb{E}(X_t^2) \sim \begin{cases} t^{\frac{2+2\alpha-\alpha^2}{1+\alpha}} & \text{if } \alpha \in (0,1) \quad \text{superdiffusive behavior} \\ t^{\frac{5}{2}-\alpha} & \text{if } \alpha \in [1, \frac{3}{2}] \quad \text{superdiffusive behavior} \\ t & \text{if } \alpha \in (\frac{3}{2}, 2) \quad \text{diffusive behavior} \end{cases}$$

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- Rare events: Big jump principle (Vezzani, Barkai, Burioni ['18])

Previous (quenched) results

Case $\alpha \in (1,2)$: finite mean, infinite variance

- Berger, Rosenthal ['13] show that if $\mu = \mathbb{E}(\zeta)$, then for P-a.e. ω

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  $\longrightarrow$ *The annealed CLT then follows trivially*
- *quenched moments convergence of $Y_n/\sqrt{n}$ to the moments of $N(0,\mu^2)$.*

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- By the ergodicity of the process seen from the particle:

$$\frac{T_n}{n} \underset{n\to\infty}{\longrightarrow} \mu, \qquad \frac{T^{-1}(t)}{t} \underset{t\to\infty}{\longrightarrow} 1/\mu, \qquad \mathbb{P}-a.s$$

Case $\alpha \in (0,1)$: infinite mean and variance

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- $\bar{S}^{(n)} = \left(\frac{S_{[nt]}}{n^{\frac{1}{2}}} \right)_{t \in \mathbb{R}^+} \xrightarrow{w} B$ invariance principle

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Theorem 2 (B., Lenci, Pène - SPA '19).

Let $\alpha \in (0,1)$. Then, under $\mathbb{P}$ and for all $k \in \mathbb{N}$ and $t_1, \ldots, t_k \in \mathbb{R}^+$,

$$(\bar{Y}^{(n)}(t_1), \ldots, \bar{Y}^{(n)}(t_k)) \xrightarrow[n \to \infty]{d} (Z \circ B(t_1), \ldots, Z \circ B(t_k))$$

i.e, the finite-dim. distributions of $\bar{Y}^{(n)}$ converge to those of $Z \circ B$.

Collision times as Random walks in random scenery

Similarly, we have

$$\bar{X}^{(n)}(t) := \frac{X_{[nt]}}{n^{1/(\alpha+1)}} \simeq \bar{\omega}^{(\sqrt{q_n})} \circ \bar{S}^{(q_n)} \circ (\bar{T}^{(n)}(t))^{-1}$$

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where $\mathcal{N}_n(k) = \sharp\{j \in \{0,\dots,n\} : [k,k+1] \subseteq [S_{j-1}, S_j]\}$

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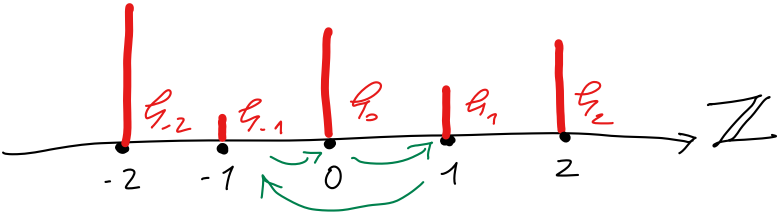
Then $(T_n)_{n\in\mathbb{N}}$ can be thought as a RW in a random scenery

Random walk in random scenery

By [Kesten, Spitzer ’79]



- $S_0 = 0 \qquad S_1 = 1 \qquad S_2 = -1 \qquad S_3 = 0$
- $T_0 = \zeta_0 \qquad T_1 = \zeta_0 + \zeta_1 \qquad T_2 = \zeta_0 + \zeta_1 + \zeta_{-1} \qquad T_3 = \zeta_0 + \zeta_1 + \zeta_{-1} + \zeta_0$

Formally $\mathcal{T}_n := \sum_{j=0}^{n} \zeta_{S_j} = \sum_{k\in\mathbb{Z}} N_n(k)\zeta_k$, $n \in \mathbb{N}$
where $N_n(k) = \sharp\{j \in \{0, \ldots, n\} \ : \ S_j = k\}$ are **local times** of S.

Convergence of RWRS: Kesten-Spitzer process

Theorem 3 (Kesten, Spitzer '79).

Let $\alpha \in (0,1)$. *Under* $\mathbb{P}$, *and taking* $n \to \infty$, *it holds*

$$\left(\frac{\mathcal{T}_{[ns]}}{n^{\frac{1+\alpha}{2\alpha}}}\right)_{s\in\mathbb{R}^+} \xrightarrow{w} \Delta \quad \text{in } D(\mathbb{R}^+, J_1)\,,$$

where $\Delta(t) = \int_{-\infty}^{+\infty} L_t(x) dZ(x)$ *Kesten-Spitzer process*

Where $L_t = (L_t(x))_{x\in\mathbb{R}}$ is the **local time** of the Browniam motion B and Z an α-stable process on $\mathbb{R}$.

Convergence of collision times

Assumption on the underlying RW: $\mathbb{E}(|\xi_1|^{2/\alpha+\varepsilon}) < \infty$.

Proposition 1 (B., Lenci, Pène - SPA '19).

Let $\alpha \in (0,1)$. *Under* $\mathbb{P}$, *and taking* $n \to \infty$, *it holds*

$$\left(\frac{T_{[ns]}}{n^{\frac{1+\alpha}{2\alpha}}} \right)_{s \in \mathbb{R}^+} \xrightarrow{w} \Delta \quad \text{in } D(\mathbb{R}^+, J_1) .$$

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Moreover, it holds the following joint convergence

Lemma 4.

Under $\mathbb{P}$*, and taking* $n \to \infty$*, it holds*

$$\left(\bar{\omega}^{(n)}, \bar{S}^{(n)}, \bar{T}^{(n)} \right) \xrightarrow{w} (Z, B, \Delta) \quad \text{in } D(\mathbb{R}, J_1) \times (D(\mathbb{R}^+, J_1))^2$$

Convergence of the main process X

Theorem 5 (B., Lenci, Pène - SPA '19).

Let $\alpha \in (0,1)$. Under $\mathbb{P}$, and taking $n \to \infty$, the finite-dimensional distributions of $\bar{X}^{(n)}$ converge to the corresponding distribution of $Z \circ B \circ \Delta^{-1}$.

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Remarks

- For $\alpha \in (0,1)$ then the processes Y and X display superdiffusive behavior with scaling exponent, resp., $1/2\alpha$ and $1/(\alpha+1)$.

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Remarks

- For $\alpha \in (0,1)$ then the processes Y and X display superdiffusive behavior with scaling exponent, resp., $1/2\alpha$ and $1/(\alpha+1)$.
- Results can not be extended to a functional limit theorem w.r.t to the Skorokhod topology as $Z \circ B$ and $Z \circ B \circ \Delta^{-1}$ have discontinuities without one-sided limits.

Case $\alpha \in (0,1)$: Proof ideas

General method: Weak convergence of Prop. 1 follows by the classic strategy: Convergence of finite dimensional distributions + tightness.

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Convergence of finite dimensional distributions are based on characteristic functions.

Key point: (T_n) behaves in the limit as a RWRS, converging to the Kesten-Spitzer process Δ.

Characteristic function of T_n

- $\bar{T}^{(n)}(s) = \frac{1}{n^{(1+\alpha)/2\alpha}} \sum_{k \in \mathbb{Z} \setminus \{0\}} \mathcal{N}_{[ns]}(k)\zeta_k$
- $\phi_\zeta(\theta) = \exp[-c_1|\theta|^\alpha(1 - \imath c_2 \mathrm{sgn}\theta)]$ *(Hyp. $\zeta \sim \alpha$-stable,)*

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$$\mathbb{E}[\exp(\imath\theta\bar{T}^{(n)}(s)) \mid S] = \prod_{k\in\mathbb{Z}\setminus\{0\}} \phi_\zeta\left(\theta\,\frac{\mathcal{N}_{[ns]}(k)}{n^{(1+\alpha)/2\alpha}}\right)$$

$$= \exp\left(-c_1|\theta|^\alpha(1 - \imath c_2 \mathrm{sgn}\theta) \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\left(\mathcal{N}_{[ns]}(k)\right)^\alpha}{n^{(1+\alpha)/2}}\right)$$

On the other hand

$$\mathbb{E}[\exp(\imath\theta\Delta(s))] = \mathbb{E}\left[\exp\left(-c_1(1-\imath c_2 \mathrm{sgn}\theta)|\theta|^\alpha \int_{\mathbb{R}} (L_s(x))^\alpha dx\right)\right]$$

and one has to show

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\left(\mathcal{N}_{[ns]}(k)\right)^\alpha}{n^{(1+\alpha)/2}} \xrightarrow{d} \int_{\mathbb{R}} (L_s(x))^\alpha dx$$

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- $\sum_{k\in\mathbb{Z}\setminus\{0\}} \mathbb{E}\left[|\mathcal{N}_n(k) - \mathbb{E}[|\xi|]N_n(k)|^\alpha\right] = o(n^{(1+\alpha)/2})$
- results and strategy implemented in [Kesten Spitzer, '79]

Conclusions

- We represent the Lévy Lorentz gas as a RW in a random scenery time, and show convergence of the collision times to Kesten Spitzer process.

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  $\Longrightarrow$ quenched diffusive behavior.

Conclusions

- We represent the Lévy Lorentz gas as a RW in a random scenery time, and show convergence of the collision times to Kesten Spitzer process.
- For $\alpha \in (1,2)$ (integrable environment) we prove in [BCLL'16] quenched CLT for discrete and continuous time process.
  $\Longrightarrow$ quenched diffusive behavior.
- For $\alpha \in (0,1)$ (non-integrable environment) we establish in [BLP'19] a functional limit theorem for discrete and continuous time.
  $\Longrightarrow$ annealed superdiffusive behavior.

Open problems

- Annealed moments
  - comparison with previous estimates and simulations;
  - comparison with persistent RW on averaged environment.
    Artuso, Cristadoro, Onofri, Radice ['18]

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- Annealed moments
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- Quenched functional convergence for $\alpha \in (0, 1)$, and Moment assumption over the underlying RW in 1D.
- What happens in dimension $D \geq 2$?
  - definition of a 2 D-*Lévy environment*;
  - comparison with 2D and 3D- models on Lévy-like environments. Buonsante, Burioni, Vezzani ['11]

Thank you for your attention!