

# Stability and uniform fluctuation estimates of Ensemble Kalman-Bucy filters

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## Synthesis joint works:

- ▶ AAP-17 (Unif. EnKBF)+ **Tugaut.**
- ▶ SIAM Control & Opt.-17 (Unif. En-EKBF)+ Kurtzmann, **Tugaut.**
- ▶ SIAM Control & Opt.-17/18 (Stability KBF)+ Bishop
- ▶ SPA-17 (Perturbation KB)+ Bishop, S. Pathiraja.
- ▶ EJP-18 (Stability EKBF)+ Kurtzmann, **Tugaut.**
- ▶ AAP-18/19 (1d-case)+ Bishop, Kamatani, Rémillard.
- ▶ Arxiv (Stochastic Riccati)+ Bishop, A. Niclas.

## Kalman-Bucy filter

### McKean-Vlasov interpretations

- 3 classes of algorithms

- Mean field/Ensemble Kalman-Bucy filter

- Dyson equation

- The 1d case

### Stability of Kalman-Bucy diffusions

- Stability of Riccati semigroup

- Stability of stochastic flows

### Uniform fluctuation estimates

- Some observations/numerical issues

- The Multivariate case

### Nonlinear models

- Extended Kalman-Bucy-filters

- Extended Ensemble Kalman-Bucy-filters

- A stability theorem

- Uniform propagation of chaos estimates

Kalman-Bucy filter

McKean-Vlasov interpretations

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# Kalman-Bucy filter

## Linear+Gaussian filtering problem

$$\begin{cases} dX_t &= A X_t dt + R^{1/2} dW_t \in \mathbb{R}^r \\ dY_t &= C X_t dt + \Sigma^{1/2} dV_t \end{cases} \rightsquigarrow \mathcal{F}_t := \sigma(Y_s, s \leq t).$$

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## Optimal $\mathbb{L}_2$ -filter = Kalman-Bucy filter

$$\hat{X}_t := \mathbb{E}(X_t | \mathcal{F}_t) \quad \text{and} \quad P_t := \mathbb{E}((X_t - \mathbb{E}(X_t | \mathcal{F}_t))(X_t - \mathbb{E}(X_t | \mathcal{F}_t))')$$

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## State estimate

$$d\hat{X}_t = A \hat{X}_t dt + P_t C' \Sigma^{-1} (dY_t - C \hat{X}_t dt)$$

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$$d\hat{X}_t = A \hat{X}_t dt + P_t C' \Sigma^{-1} (dY_t - C \hat{X}_t dt)$$

with the gain given by the matrix Riccati equation

$$\partial_t P_t = \text{Ricc}(P_t) := AP_t + P_t A' - P_t S P_t + R \quad \text{with} \quad S := C' \Sigma C$$

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# Reformulation $\rightsquigarrow$ Nonlinear Kalman-Bucy diffusion

$\iff$  McKean-Vlasov type diffusions  $\bar{X}_t$  such that

$$\eta_t := \text{Law}(\bar{X}_t \mid \mathcal{F}_t) = \mathcal{N}[\hat{X}_t, P_t]$$

$\rightsquigarrow$  Interacting with their conditional mean and covariance matrices

$$\mathcal{P}_{\eta_t} = \eta_t [(e - \eta_t(e))(e - \eta_t(e))'] \quad \text{with} \quad e(x) := x.$$

### 3 classes of McKean-Vlasov type diffusions

#### 1) "Vanilla EnKF" ( $\rightsquigarrow$ discrete time - Evensen 94)

$$d\bar{X}_t = A \bar{X}_t dt + R^{1/2} d\bar{W}_t + \mathcal{P}_{\eta_t} C' \Sigma^{-1} \left[ dY_t - \left( C \bar{X}_t dt + \Sigma^{1/2} d\bar{V}_t \right) \right]$$

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3) Pure transport equation (Reich-Cotter 13):

$$d\bar{X}_t = A \bar{X}_t dt$$

$$+ \frac{1}{2} (R - \mathcal{P}_{\eta_t} S \mathcal{P}_{\eta_t}) \mathcal{P}_{\eta_t}^{-1} (\bar{X}_t - \eta_t(e)) dt + \mathcal{P}_{\eta_t} C' \Sigma^{-1} [dY_t - C \eta_t(e) dt]$$

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⊕ Many others, adding  $\mathcal{Q}_{\eta_t} \mathcal{P}_{\eta_t}^{-1} (\bar{X}_t - \eta_t(e)) dt$  for any  $\mathcal{Q}'_{\eta_t} = -\mathcal{Q}_{\eta_t}$ .

# The Ensemble Kalman-Bucy filter

**(Case 1) Mean field interpretation  $\rightsquigarrow N + 1$  interacting diffusions**

$$d\xi_t^i = A \xi_t^i dt + R^{1/2} d\bar{W}_t^i + p_t C' \Sigma^{-1} \left[ dY_t - \left( C \xi_t^i dt + \Sigma^{1/2} d\bar{V}_t^i \right) \right]$$

**with the rescaled particle covariance matrices**

$$p_t := \left( 1 + \frac{1}{N} \right) P_{\eta_t^N} = \frac{1}{N} \sum_{1 \leq i \leq N+1} (\xi_t^i - m_t) (\xi_t^i - m_t)'$$

**and the empirical measures**

$$\eta_t^N := \frac{1}{N+1} \sum_{1 \leq i \leq N+1} \delta_{\xi_t^i} \quad \text{and the sample mean} \quad m_t := \frac{1}{N+1} \sum_{1 \leq i \leq N+1} \xi_t^i.$$

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*where is the Riccati equation?*

# Th1: The EnKF equations

## The EnKF equation

$$dm_t = A m_t dt + p_t C' \Sigma^{-1} (dY_t - C m_t dt) + \frac{1}{\sqrt{N+1}} d\bar{M}_t$$

with an  $r$ -dimensional martingale  $\bar{M}_t = (\bar{M}_t(k))_{1 \leq k \leq r}$  with angle-brackets

$$\partial_t \langle \bar{M} | \otimes | \bar{M} \rangle_t = U + p_t V p_t.$$

## With

- 1)  $(U, V) = (R, S)$     2)  $(U, V) = (R, 0)$     3)  $(U, V) = (0, 0)$



## The particle/ensemble Riccati equation

$$dp_t = \text{Ricc}(p_t) dt + \frac{1}{\sqrt{N}} dM_t$$

Symmetric matrix-valued martingale  $M_t = (M_t(k, l))_{1 \leq k, l \leq r}$

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Angle brackets = the Wick-type formula  $((\cdot \otimes \cdot)^\sharp := \text{entrywise})$

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Orthogonality property

$$\forall 1 \leq k, l, l' \leq r \quad \langle M(k, l), \overline{M}(l') \rangle_t = 0.$$

In terms of random matrices with  $\epsilon := \frac{2}{\sqrt{N}}$

$\mathcal{W}_t = (\mathcal{W}_t(i, j))_{1 \leq i, j \leq r}$  independent Brownian motions

↓

$$dp_t \stackrel{law}{=} [Ap_t + p_t A' + R - p_t S p_t] dt + \epsilon \left( p_t^{1/2} d\mathcal{W}_t (U + p_t V p_t)^{1/2} \right)_{sym}$$

▶  $S = 0 = V \rightsquigarrow$  **Wishart process**

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$p_t \rightsquigarrow$  non colliding eigenvalues  $\lambda_r(t) < \dots < \lambda_2(t) < \lambda_1(t)$

satisfying the **Dyson equation**

$$d\lambda_i(t) =$$

$$\left[ 2\alpha\lambda_i(t) + \beta - \lambda_i(t)^2\gamma + \frac{\epsilon^2}{4} \sum_{j \neq i} \frac{\lambda_i(t) + \lambda_j(t)}{\lambda_i(t) - \lambda_j(t)} \right] dt + \epsilon \sqrt{\lambda_i(t)} dW_t^i$$

## The 1d case $\rightsquigarrow$ Closed form Riccati semigroups

**Deterministic Riccati  $P_t$  on  $\mathbb{R}_+$ :**  $\text{Ricc}(\varpi_{\pm}) = 0$  for

$$S\varpi_- := A - \lambda/2 < 0 < S\varpi_+ := A + \lambda/2$$

with

$$\lambda = 2\sqrt{A^2 + RS}$$

$\Downarrow$

$\forall t \geq v > 0$

$$\left[ |P_t - \varpi_+| \vee \exp\left(2 \int_0^t [A - P_s S] ds\right) \right] \leq c_v \exp(-\lambda t)$$



**Stochastic Riccati flow**  $p_t \in \mathbb{R}_+$  with  $\epsilon = 2/\sqrt{N}$ :

$$dp_t \stackrel{\text{law}}{=} \text{Ricc}(p_t)dt + \epsilon \sqrt{p_t(U + p_t V p_t)} dW_t$$

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**Reversible measures**  $\pi_\epsilon(dx)$  on  $\mathbb{R}_+$ :

►  $U \wedge V > 0 \rightsquigarrow$  **Heavy tails**

$$\propto \frac{x^{\frac{2}{\epsilon^2} \frac{R}{U} - 1}}{[U + Vx^2]^{1 + \frac{1}{\epsilon^2} (\frac{R}{U} + \frac{S}{V})}} \exp \left[ \frac{4}{\epsilon^2} \frac{A}{\sqrt{UV}} \tan^{-1} \left( x \sqrt{\frac{V}{U}} \right) \right] dx.$$

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▶  $U > V = 0 \rightsquigarrow$  **Gaussian-type tails**

$$\propto x^{\frac{2}{\epsilon^2} \frac{R}{U} - 1} \exp \left[ -\frac{S}{U\epsilon^2} \left( x - 2 \frac{A}{S} \right)^2 \right] dx.$$

## Stability Markov transition semigroup $\mathcal{P}_t^\epsilon$ (of $p_t$ )

**Th** [+ Bishop, Kamatani, Rémillard Arxiv-17]  $\forall A, R \wedge S > 0$

- $\exists \zeta, \epsilon_0 > 0$  and some Wasserstein distance  $\mathbb{D}$  s.t. for any  $0 \leq \epsilon \leq \epsilon_0$

$$\mathbb{D}(\mu_1 \mathcal{P}_t^\epsilon, \mu_2 \mathcal{P}_t^\epsilon) \leq \exp(-\lambda (1 - \epsilon^2 \zeta) t) \mathbb{D}(\mu_1, \mu_2)$$

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- $\forall n \geq 1 \exists \zeta_n, \epsilon_n > 0$  for any  $0 \leq \epsilon \leq \epsilon_n$

$$\mathbb{E} \left[ \exp \left[ n \int_0^t (A - p_s S) ds \right] \right]^{1/n} \leq c_Q \exp(-\lambda (1 - \epsilon^2 \zeta_n) t)$$

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## Some extensions

**Case 2:** Poincaré inequalities (and  $\mathbb{L}_2(\pi_\epsilon)$ -contractions), ...

## Consequences

Uniform estimates for state estimates + particle Riccati diffusions, ...

Kalman-Bucy filter

McKean-Vlasov interpretations

Stability of Kalman-Bucy diffusions

Stability of Riccati semigroup

Stability of stochastic flows

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## Multivariate KB : Observability + Controllability

$$\begin{aligned} & d(\widehat{X}_t - X_t) \\ &= (\mathbf{A} - \mathbf{P}_t \mathbf{S}) (\widehat{X}_t - X_t) dt - \mathbf{R}^{1/2} d\mathbf{W}_t + \mathbf{P}_t \mathbf{C}' \boldsymbol{\Sigma}^{-1/2} d\mathbf{V}_t \end{aligned}$$



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**Steady state:**  $\exists! P_\infty > 0$  s.t.  $\text{Ric}(P_\infty) = 0$  and spectral abscissa

$$\varsigma(\mathbf{A} - \mathbf{P}_\infty \mathbf{S}) := \max \{ \text{Re}(\lambda) : \lambda \in \text{Spec}(\mathbf{A} - \mathbf{P}_\infty \mathbf{S}) \} < 0$$

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**STABLE EVEN WHEN  $A$  is unstable.**

↪ SIAM Control & Opt.-17  $\oplus$  Arxiv-18 (+ Bishop )  
*Review on the stability of Kalman-Bucy filters and Riccati matrix semigroups  $\oplus$  Floquet representation of exponential semigroups*

# Floquet representations

$$P_t = \phi_t(P_0) \quad \text{flow of the Riccati equation} \quad \partial_t P_t = \text{Ricc}(P_t)$$



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$$E_{s,t}(P) = \exp \int_s^t (A - \phi_u(P)S) du$$

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**in the sense that (with  $E_{t,t}(P) = Id$ )**

$$\partial_t E_{s,t}(P) = (A - \phi_t(P)S)E_{s,t}(P) \quad \text{and} \quad \partial_s E_{s,t}(P) = -E_{s,t}(P)(A - \phi_s(P)S)$$

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**Nb.:**

$$P = P_\infty \implies E_{s,t}(P_\infty) = e^{(t-s)(A - P_\infty S)} \quad \text{with} \quad A - P_\infty S \quad \text{stable}$$

## Floquet representations 2/2

**Theo.:** (+ Bishop - Arxiv-18)

$$E_t(P) := \exp \int_0^t (\mathbf{A} - \phi_s(\mathbf{P})\mathbf{S}) ds = e^{t(\mathbf{A} - P_\infty \mathbf{S})} \mathbb{C}_t(P)^{-1}$$

with

$$\sup_{t \geq 0} \|\mathbb{C}_t(P)^{-1}\| \leq c (1 + \|Q\|)$$

**Cor.:**

$$\|\phi_t(P_1) - \phi_t(P_2)\| \leq \|e^{t(\mathbf{A} - P_\infty \mathbf{S})}\| (1 + \|P_1\|^2 + \|P_2\|^2) \|P_1 - P_2\|$$

⊕ same type of estimates for the time varying linear process

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- ▶ **Case 3** = *Pure transport*  $\rightsquigarrow$  simple estimates using Lipschitz and contraction inequalities w.r.t. initial conditions

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$(m_t, X_t, p_t) = (\text{sample mean, true signal, sample covariance})$

↓

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- ▶ Always under-biased

$$\forall t > 0 \quad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) < P_t$$



## Multivariate case

**Hyp 1:**  $S > 0 \rightsquigarrow$  up a change of basis

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+ Uniform CLT rates + Bias-Taylor type expansions + Robustness and Perturbations analysis (inflation, masking, shrinkage, projections, ...)

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- Extended Kalman-Bucy-filters

- Extended Ensemble Kalman-Bucy-filters

- A stability theorem

- Uniform propagation of chaos estimates

# Nonlinear models

## Extended Kalman-Bucy-filters

$$d\hat{X}_t = A(\hat{X}_t) dt + P_t C' \Sigma^{-1} [dY_t - C\hat{X}_t dt]$$

with the "stochastic" Riccati equation:

$$\partial_t P_t = \partial A(\hat{X}_t) P_t + P_t \partial A(\hat{X}_t)' + R - P_t S P_t$$

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## McKean-Vlasov interpretation

$$\begin{aligned} d\bar{X}_t = & \mathcal{A}(\bar{X}_t, \eta_t(e)) dt + R^{1/2} d\bar{W}_t \\ & + \mathcal{P}_{\eta_t} C' R_2^{-1} \left[ dY_t - \left( C\bar{X}_t dt + \Sigma^{1/2} d\bar{V}_t \right) \right] \end{aligned}$$

with the drift function

$$\mathcal{A}(x, m) := A[m] + \partial A[m] (x - m).$$

# Extended Ensemble Kalman-Bucy-filters

## En-EKF = Mean field particle model

$$d\xi_t^i = \mathcal{A}(\xi_t^i, m_t) dt + R^{1/2} d\bar{W}_t^i + p_t C' \Sigma^{-1} \left[ dY_t - \left( C \xi_t^i dt + \Sigma^{1/2} d\bar{V}_t^i \right) \right]$$

with the sample means  $m_t$  and covariance matrices  $p_t$  and the drift

$$\mathcal{A}(\xi_t^i, m_t) := A[m_t] + \underbrace{\frac{\partial A[m_t]}{\partial m_t} (\xi_t^i - m_t)}_{\text{Repulsion/Attraction w.r.t. } m_t}$$



# Some illustrations

## Langevin type signal processes

$$R = \sigma^2 Id \quad \text{and} \quad (A, \partial A) = (-\partial\mathcal{V}, -\partial^2\mathcal{V})$$

## Non quadratic potential ( $q \in \mathbb{R}^r, Q_1, Q_2 \geq 0$ )

$$\mathcal{V}(x) = \frac{1}{2} \langle Q_1 x, x \rangle + \langle q, x \rangle + \frac{1}{3} \langle Q_2 x, x \rangle^{3/2}$$

## Interacting diffusion gradient flows

$$\mathcal{V}(x) = \sum_{1 \leq i \leq r} \mathcal{U}_1(x_i) + \sum_{1 \leq i \neq j \leq r} \mathcal{U}_2(x_i, x_j)$$

for some convex confining potential  $\mathcal{U}_i : \mathbb{R}^i \mapsto [0, \infty[$

# Regularity conditions

**Full observation  $S = s Id$  and**

$$-\lambda_{\partial A} := \sup_{x \in \mathbb{R}^r} \lambda_{\max}(\partial A(x) + \partial A(x)') < 0$$

$$\|\partial A(x) - \partial A(y)\| \leq \kappa_{\partial A} \|x - y\|$$

**Examples: Langevin signal-diffusion**

$$(\lambda_{\partial A}, \kappa_{\partial A}) = \beta \left( 2^{-1} \lambda_{\min}(Q_1), 2 \lambda_{\max}^{3/2}(Q_2) \right).$$

*more generally  $\partial^2 \mathcal{V} \geq \nu Id \oplus$  Lipschitz condition*

# Stability theorem

$(\bar{X}_t, \bar{Z}_t) :=$  McKean-Vlasov starting at  $(\bar{X}_0, \bar{Z}_0)$

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**Theo** [+Kurtzmann-Tugaut]

When  $\lambda_{\partial A}$  is sufficiently large we have

$$\mathbb{W}_2(\text{Law}(\bar{X}_t), \text{Law}(\bar{Z}_t)) \leq c \exp[-t \lambda] \quad \text{for some } \lambda > 0.$$

$\exists$  *more explicit description in terms of*  $(R, S, \kappa_{\partial A})$ .

# Propagation of chaos

$$\mathbb{P}_t^N := \text{Law}(m_t, p_t) \quad \mathbb{P}_t := \text{Law}(\widehat{X}_t, P_t)$$

and

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**Theo** [+Kurtzmann-Tugaut]

When  $\lambda_{\partial A}$  is sufficiently large,  $\exists \beta \in ]0, 1/2]$  s.t.

$$\sup_{t \geq 0} \mathbb{W}_2(\mathbb{P}_t^N, \mathbb{P}_t) \vee \sup_{t \geq 0} \mathbb{W}_2(\mathbb{Q}_t^N, \mathbb{Q}_t) \leq c N^{-\beta}$$

as soon as  $\text{tr}(P_0)$  is not too large and  $N$  large enough...