Bayesian methods for excursion set estimation

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The framework

In this talk we focus on the problem of determining the set

$$\Gamma^\star = \{x \in D : f(x) \in T\} = f^{-1}(T)$$

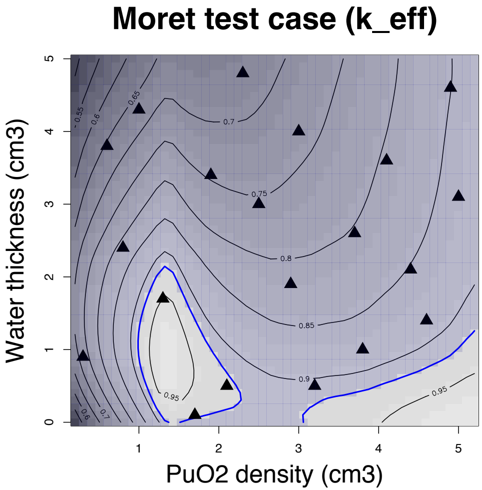
where $D \subset \mathbb{R}^d$ is compact, $f : D \longrightarrow \mathbb{R}^k$ is continuous, $T \subset \mathbb{R}^k$.

Here: $k = 1$, and $T = (-\infty, t]$ for a fixed $t \in \mathbb{R}$.
$\Gamma^\star = \{x \in D : f(x) \leq t\}$ is denoted the excursion set of f below t.

Objective

Estimate $\Gamma^\star$ and quantify uncertainty on it when f is evaluated only at a few points $\mathbf{X}_n = \{x_1, \ldots, x_n\} \subset D$.

Some applications



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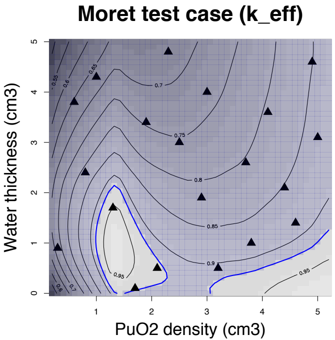
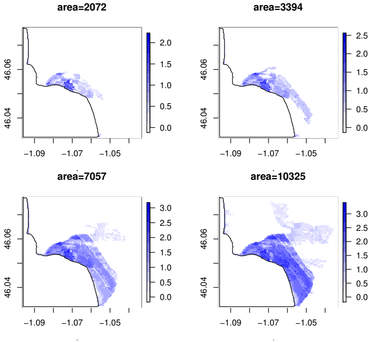


Coastal flooding



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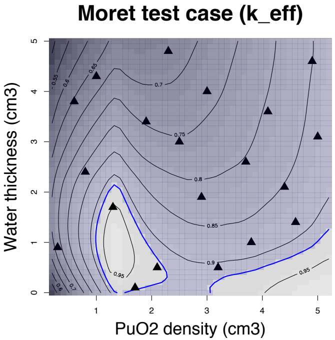
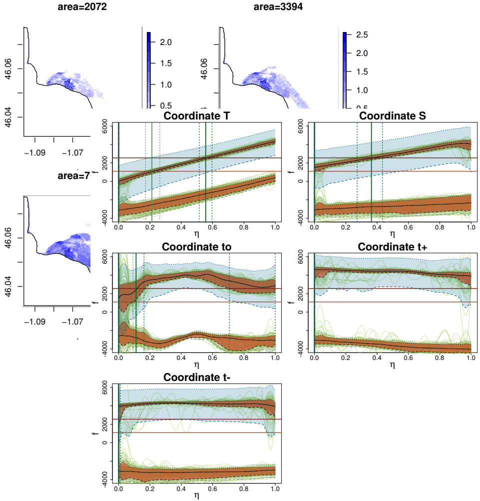




Selected references

This problem has been tackled in many previous works such as

Sacks, J., Welch, W. J., Mitchell, T. J., and Wynn, H. P. (1989). Design and analysis of computer experiments. Statistical Science, 4(4):409-435.

Ranjan, P., Bingham, D., and Michailidis, G. (2008). *Sequential experiment design for contour estimation from complex computer codes.* Technometrics, 50(4):527-541.

Bect, J., Ginsbourger, D., Li, L., Picheny, V., and Vazquez, E. (2012). *Sequential design of computer experiments for the estimation of a probability of failure.* Stat. Comput., 22 (3):773-793.

Chevalier, C., Bect, J., Ginsbourger, D., Vazquez, E., Picheny, V., and Richet, Y. (2014). *Fast kriging-based stepwise uncertainty reduction with application to the identification of an excursion set.* Technometrics, 56(4):455-465.

Bolin, D. and Lindgren, F. (2015). *Excursion and contour uncertainty regions for latent Gaussian models.* JRSS: B, 77(1):85-106.

among many others

Main questions tackled in this work

- How to use meta-models, in particular Gaussian processes to estimate the set?

  (Vorob'ev expectation, conservative estimates)

- How to sequentially reduce the uncertainty on such estimates?

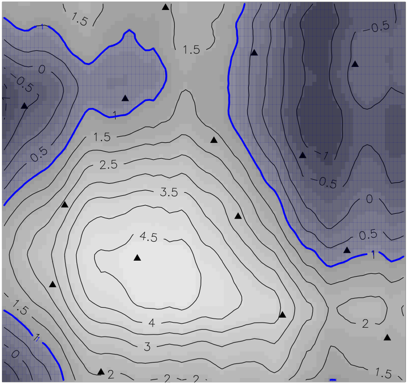
  (SUR strategies for conservative estimates)

- How to visualize the excursion set in high dimensions?

  (Profile extrema functions)

The framework: an example

Gaussian random field realization



Function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

- $D = [0, 1] \times [0, 1]$
- continuous.

Evaluated at $n = 15$ points $\mathbf{X}_n = (x_1, \ldots, x_n)$ (black triangles) with values $\mathbf{f}_n = (f(x_1), \ldots, f(x_n))$.

Objective: estimate $\Gamma^\star = \{x \in D : f(x) \leq t\}$ and evaluate the uncertainty of the estimate.

Bayesian approach

Bayesian framework: $\mathbf{f}$ *is seen as one realization of a Gaussian random field (GRF)* $(Z_x)_{x\in D}$ with prior mean m and covariance kernel k.

Given the function evaluations $\mathbf{f}_n$ the posterior field has a Gaussian distribution

$$Z \mid (Z(\mathbf{X}_n) = \mathbf{f}_n)$$

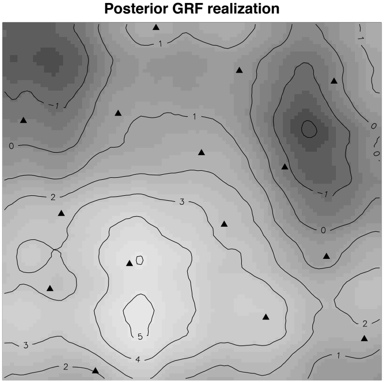
with mean and covariance kernel

$$m_n(x) = m(x) + k(x, \mathbf{X}_n)k(\mathbf{X}_n, \mathbf{X}_n)^{-1}(\mathbf{f}_n - m(\mathbf{X}_n))$$
$$k_n(x, y) = k(x, y) - k(x, \mathbf{X}_n)k(\mathbf{X}_n, \mathbf{X}_n)^{-1}k(\mathbf{X}_n, y)$$

$\Gamma^\star$ is a realization of $\Gamma = \{x \in D : Z_x \leq t\} = Z^{-1}((-\infty, t])$

A prior on the space of functions

Assume: f realization of $(Z_x)_{x\in D}$, Gaussian Random Field (GRF)



Prior: $(Z_x)_{x\in D}$ with

- a.s. continuous paths;
- Matérn covariance kernel k ($\nu = 3/2$);
- constant mean function m.

Given $n = 15$ evaluations $\mathbf{f}_n$ at $\mathbf{X}_n$

The closed form fomulas for m_n and k_n allow us to generate realizations of the **posterior field**.

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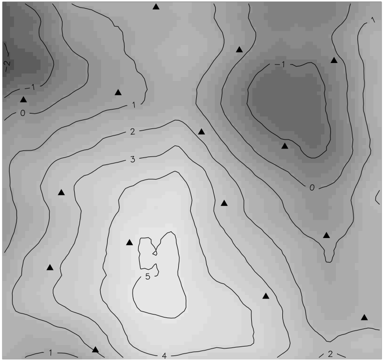
Posterior GRF realization



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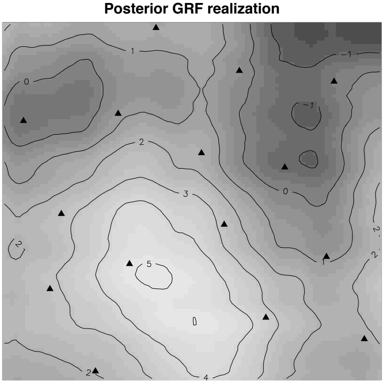

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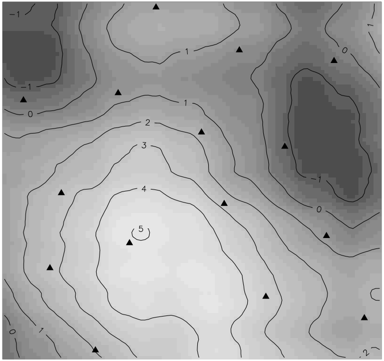

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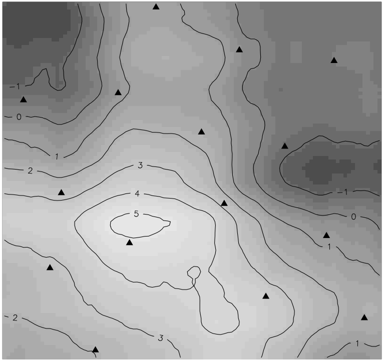

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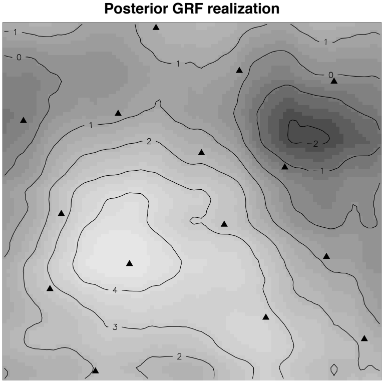

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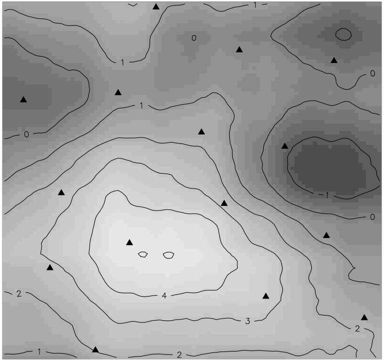

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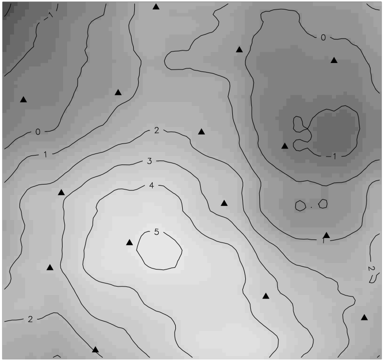

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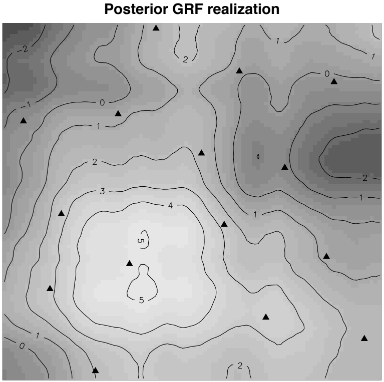

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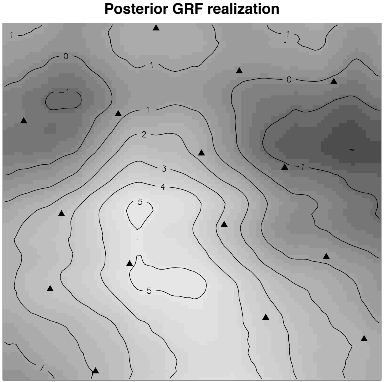

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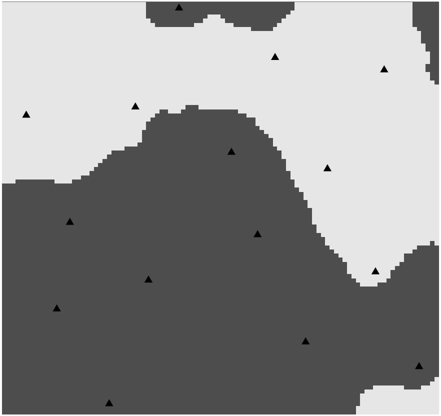
Distribution of excursion sets

The posterior field induces a posterior distribution on excursion sets.



Note that

- Z continuous paths;
- $(-\infty, t]$ is a closed set.

The set $\Gamma = \{x \in D : Z_x \leq t\}$ is a random closed set.

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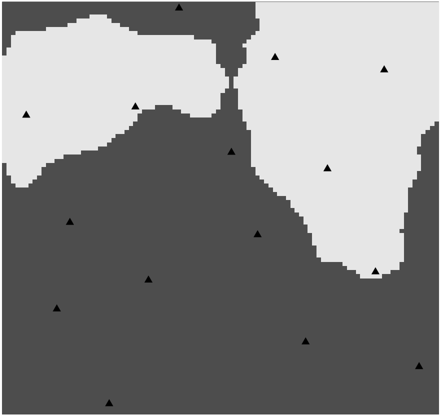

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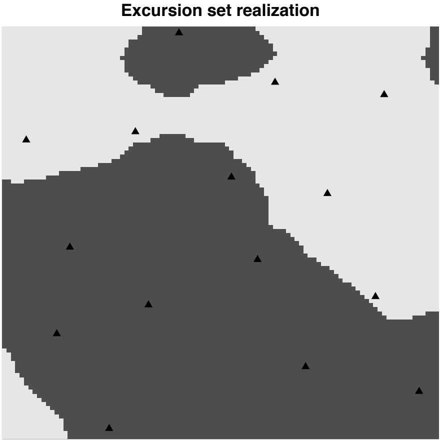

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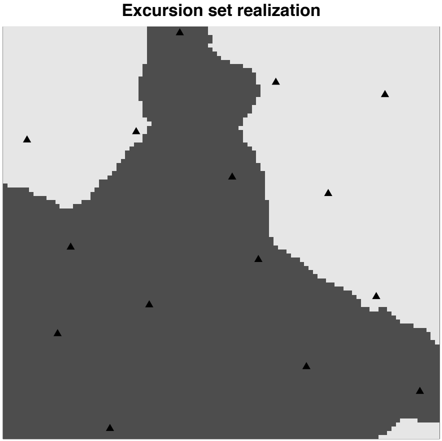

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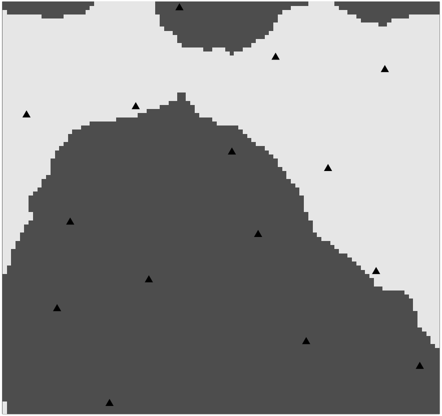

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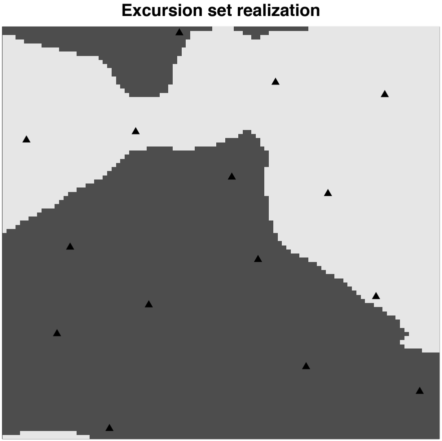

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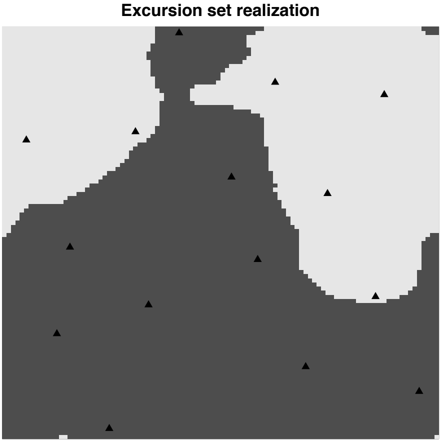

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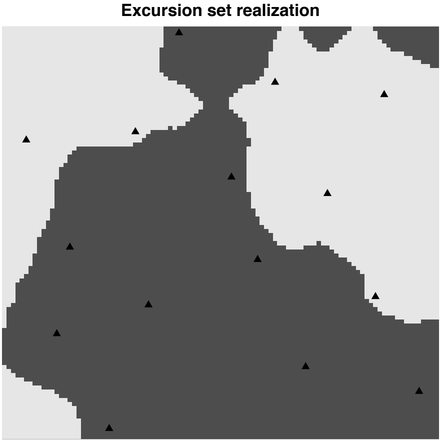

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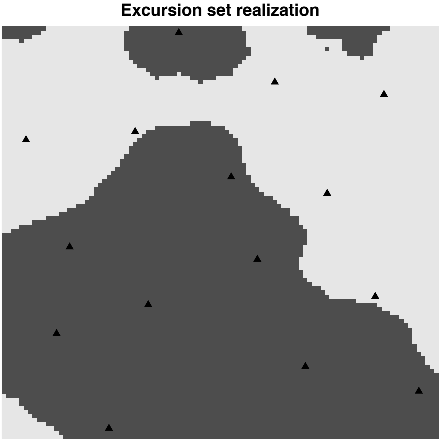

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How to summarize the distribution on sets?

Here we consider the computational challenges associated with

Estimates for Γ^* with

- **Expectations** of random closed sets[1], (*Vorob'ev expectation*).
- **Conservative estimates**, based on Vorob'ev quantiles.

Reduce the uncertainties on excursion set estimates.

Visualization of the excursion set with profile extrema functions.

1. for more definitions of expectation see Molchanov, I. (2005). Theory of Random Sets. Springer.

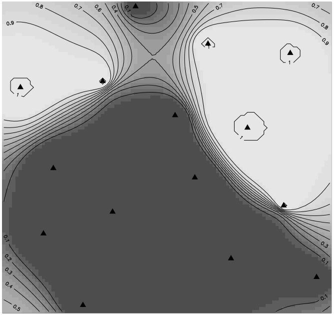
Outline

Outline

Vorob'ev quantiles

The function $p_n : x \in D \rightarrow p_n(x) = P_n(x \in \Gamma) \in [0,1]$ is the **coverage function of** Γ, where $P_n(\cdot) = P(\cdot \mid Z_{\mathbf{X}_n} = \mathbf{f}_n)$.

Coverage probability function



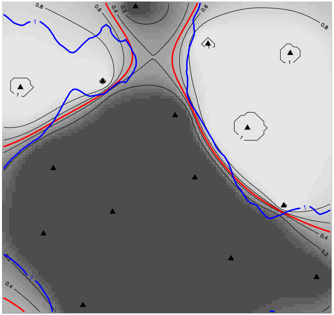
In the Gaussian case

- $p_n(x) = \Phi\left(\frac{t - m_n(x)}{\sqrt{k_n(x,x)}}\right)$
- creates a family of set estimates $Q_\rho = \{x \in D : p_n(x) \geq \rho\}$
- high ρ, then points in Q_ρ have high marginal probability.

Vorob'ev quantiles

The function $p_n : x \in D \to p_n(x) = P_n(x \in \Gamma) \in [0,1]$ is the **coverage function of** Γ, where $P_n(\cdot) = P(\cdot \mid Z_{\mathbf{X}_n} = \mathbf{f}_n)$.

Vorob'ev expectation



In the Gaussian case

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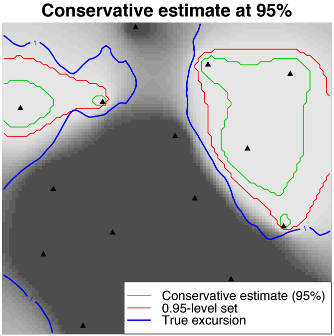
- creates a family of set estimates $Q_\rho = \{x \in D : p_n(x) \geq \rho\}$
- high ρ, then points in Q_ρ have high marginal probability.
- **Vorob'ev expectation:** choose $Q_{\widetilde{\rho}}$ such that $\mu(Q_{\widetilde{\rho}}) = \mathbb{E}[\mu(\Gamma)]$. See Chevalier et al (2013).

A **conservative estimate** of Γ^* is

$$C_{\Gamma,n} = Q_{\rho^*} \text{ where } \rho^* \in \arg\max_{\rho \in [0,1]} \{\mu(Q_\rho) : P_n(Q_\rho \subset \Gamma) \geq \alpha\}$$



- joint confidence statement on the set estimate;
- method introduced for Gauss Markov random fields, expensive to compute otherwise.
- computational bottleneck: estimation of $P_n(Q_\rho \subset \Gamma)$.

Bolin, D. and Lindgren, F. (2015). *Excursion and contour uncertainty regions for latent Gaussian models.* JRSS: B, 77(1):85-106.

The computation of conservative estimates

The family of sets Q_ρ is *nested*, therefore we can obtain $C_{\Gamma,n}$ with a **dichotomy on the level** ρ.

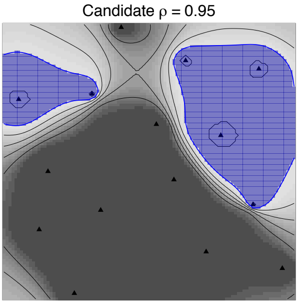
At each iteration of the dichotomy we need to compute

$$P_n(Q_\rho \subset \Gamma) = P_n(Q_\rho \subset \{Z_x \leq t\}) = P_n(Z_{e_1} \leq t, \dots, Z_{e_k} \leq t),$$

where $E = \{e_1, \dots, e_k\}$ is a the discretization of Q_ρ.

- randomized quasi Monte Carlo integration by Genz et al. : *(Fast, reliable, dimension dependent, available only $k < 1000$)*
- standard Monte Carlo. *(dimension independent, many samples for low variance)*

A quasi Monte Carlo algorithm for orthant probabilities



$$P_n(Q_\rho \subset \{Z_x \leq t\}) =$$
$$P_n(Z_{e_1} \leq t, \dots, Z_{e_k} \leq t) =$$
$$1 - P_n(\max_{x \in E} Z_x > t) = 1 - p$$

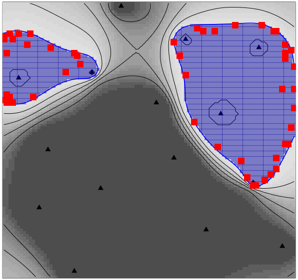
Main idea: $p = P_n(\max_E Z_x > t) = p_q + (1 - p_q) R_q$, where

$$p_q = P_n(\max_{E_q} Z_x > t), \qquad R_q = P_n(\max_{E \setminus E_q} Z_x > t \mid \max_{E_q} Z_x \leq t).$$

Genz algorithm (QRSVN) — Monte Carlo methods

A quasi Monte Carlo algorithm for orthant probabilities

Candidate $\rho = 0.95$



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Genz algorithm (QRSVN) Monte Carlo methods More

$$\widehat{p_q} = 0.47 \qquad \widehat{R_q} = 0.42 \qquad \Rightarrow \widehat{p} = 0.69$$

Computation of the remainder

$R_q = P_n(\max_{E \setminus E_q} Z_x > t \mid \max_{E_q} Z_x \leq t)$

Standard Monte Carlo:

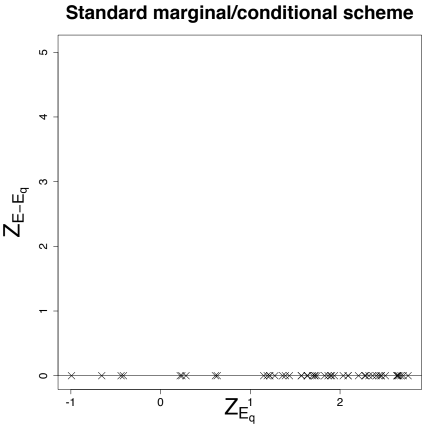
1. draw realizations $z_1^q, \dots, z_s^q$ from $Z_{E_q} \mid \max_{E_q} Z_x \leq t$;
2. for each z_i^q, draw a realization from $Z_{E \setminus E_q} \mid Z_{E_q} = z_i^q$;
3. Estimate R_q with $R_q^{\mathrm{MC}} = \frac{1}{s} \sum_{i=1}^{s} \mathbf{1}_{\max(Z_{E \setminus E_q}(\omega_i) \mid Z_{E_q} = z_i^q) > t}$

The cost of step 1 is higher than the cost of step 2.

At fixed computational budget we reduce the variance of R_q^{MC} exploiting this difference with **asymmetric nested Monte Carlo**.

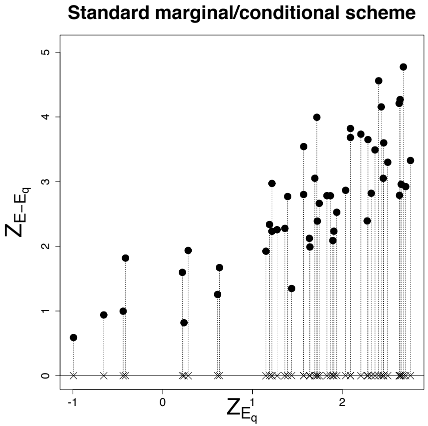
Computation of the remainder

At fixed computational budget we reduce the variance of R_q^{MC} drawing many realizations of $Z_{E \setminus E_q} \mid Z_{E_q} = z_i^q$ for each z_i.



Computation of the remainder

At fixed computational budget we reduce the variance of R_q^{MC} drawing many realizations of $Z_{E \setminus E_q} \mid Z_{E_q} = z_i^q$ for each z_i.



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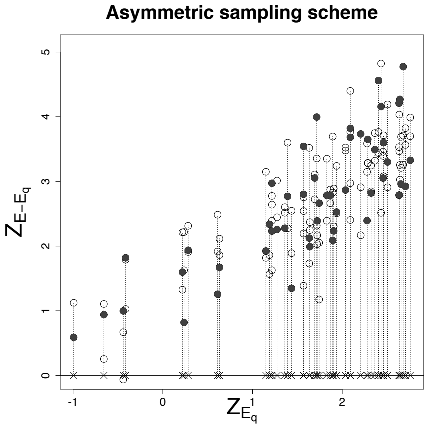


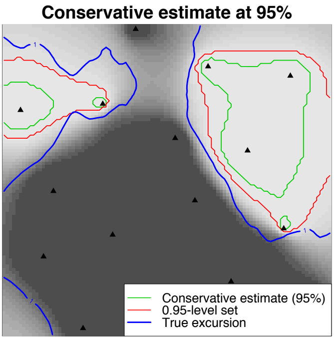
Computation of the remainder: asymmetric nested MC

$R_q = P_n(\max_{E \setminus E_q} Z_x > t \mid \max_{E_q} Z_x \leq t)$

1. *draw realizations* $z_1^q, \dots, z_s^q$ *from* $Z_{E_q} \mid \max_{E_q} Z_x \leq t$;
2. *for each* z_i^q, **draw** $m^* > 1$ **samples** from $Z_{E \setminus E_q} \mid Z_{E_q} = z_i^q$;
3. $R_q^{\text{anMC}} = \frac{1}{s} \frac{1}{m^*} \sum_{i=1}^{s} \sum_{j=1}^{m^*} \mathbf{1}_{\max(Z_{E \setminus E_q}(\omega_{i,j}) \mid Z_{E_q} = z_i^q) > t}$

$\mathrm{var}(R_q^{\text{anMC}})$ **is optimally reduced if:** $m^* = \sqrt{\frac{(\alpha + c)B}{\beta(A - B)}}$,
where $A = \mathrm{var}(\mathbf{1}_{\max(Z_{E \setminus E_q} \mid Z_{E_q}) > t})$, $B = \mathbb{E}\left[\mathrm{var}(\mathbf{1}_{\max(Z_{E \setminus E_q} \mid Z_{E_q}) > t} \mid \max_{E_q} Z_x \leq t)\right]$ and α, β, c system dependent constants. More

Comparison with standard Monte Carlo



Full discretization: grid 100×100

Dimension of candidate sets between 971 and 2370.

Time for equivalent estimates:

- Full MC: 1520 seconds;
- GMC: 200 seconds;
- GanMC: 136 seconds.

A., D. and Ginsbourger D. (2018). *Estimating orthant probabilities of high dimensional Gaussian vectors with an application to set estimation*. J. Comput. Graph. Statist., 27(2):255267. hal-01289126

Nuclear safety test case

Outline

Sequential strategies for uncertainty reduction

Sequential design of experiments: sequentially evaluate f at points that minimize a specific acquisition function.

Stepwise uncertainty reduction strategies: find a sequence of evaluation points $X_1, X_2, \ldots$ that optimally reduces the expected uncertainty on the future estimate, i.e. given an initial design $\mathbf{X}_n$, select

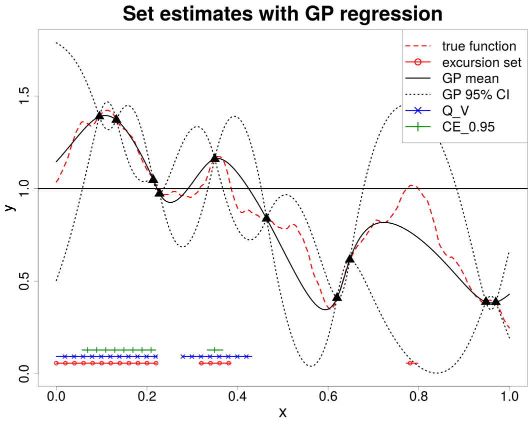
$$X_{n+1} \in \arg \min_{x_{n+1} \in D} \mathbb{E}_n[H_{n+1} \mid X_{n+1} = x_{n+1}]$$

Issues:

- define appropriate uncertainty functions for set estimates;
- minimize the expected uncertainty.

Bect, J., Bachoc, F., and Ginsbourger, D. (2018+). A supermartingale approach to Gaussian process based sequential design of experiments. hal-01351088.

Example



- Initial design, $n = 10$, maximin LHS
- constant mean function
- Matérn $\nu = 3/2$ covariance
- threshold $t = 1$
- Vorob'ev and conservative estimates.

How to reduce the uncertainty on the estimate?

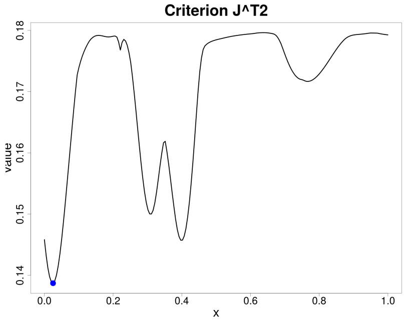
Uncertainty function(s): many possible definitions

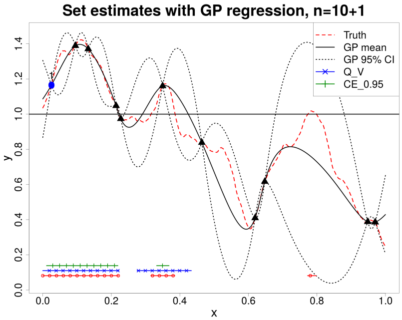
$$H_n(\rho) = \mathbb{E}_n[\mu(\Gamma \Delta Q_{n,\rho})], \quad \Gamma \Delta Q_{n,\rho} = \Gamma \setminus Q_{n,\rho} \cup Q_{n,\rho} \setminus \Gamma,$$
$$H_n^{\mathrm{T2}}(\rho_n^\alpha) = \mathbb{E}_n[\mu(\Gamma \setminus Q_{n,\rho_n^\alpha})],$$

For each uncertainty function, we define the following **SUR criteria**

$$J_n(x) = \mathbb{E}_n[H_{n+1}(\rho) \mid X_{n+1} = x] = \mathbb{E}_n[\mu(\Gamma \Delta Q_{n+1,\rho}) \mid X_{n+1} = x]$$
$$J_n^{\mathrm{T2}}(x) = \mathbb{E}_n[H_{n+1}^{\mathrm{T2}}(\rho_n^\alpha) \mid X_{n+1} = x] = \mathbb{E}_n[\mu(\Gamma \setminus Q_{n+1,\rho_n^\alpha}) \mid X_{n+1} = x],$$

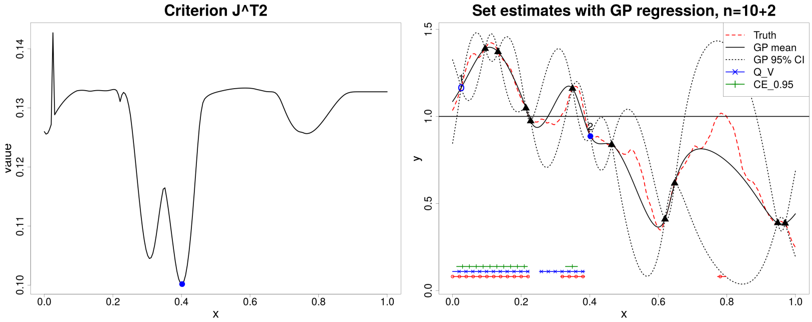
Example: T2 criterion





More

Example: T2 criterion



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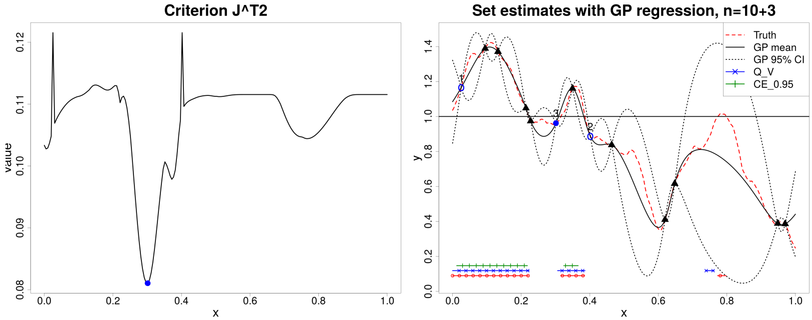

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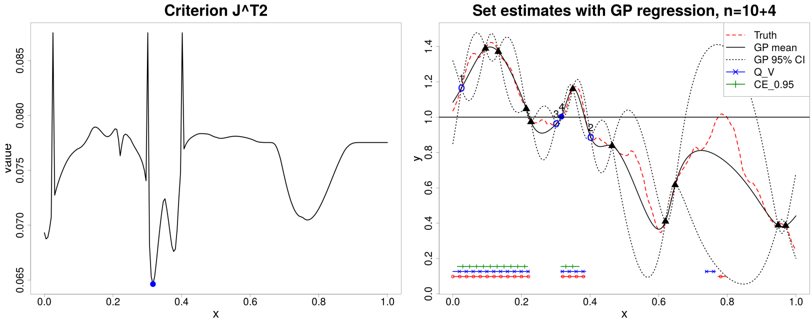

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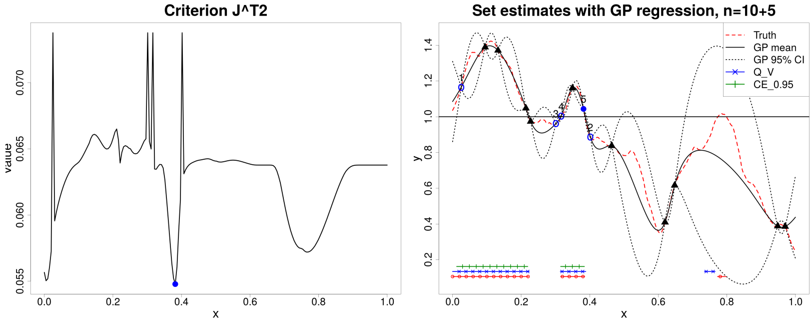

More

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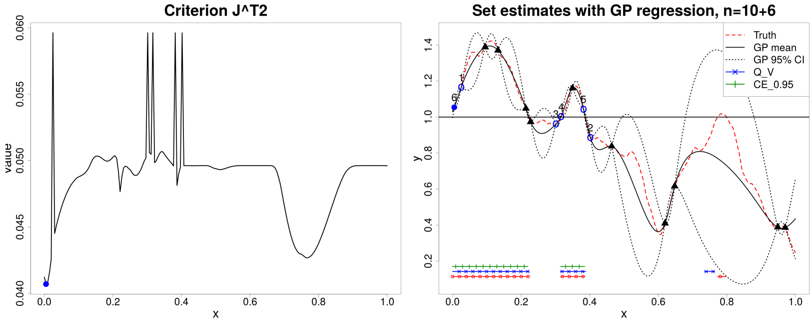

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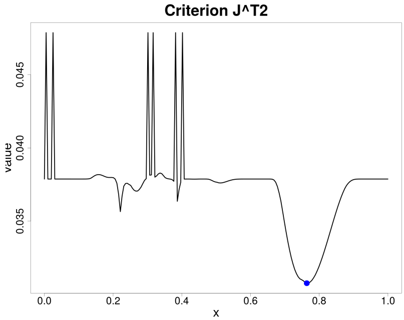

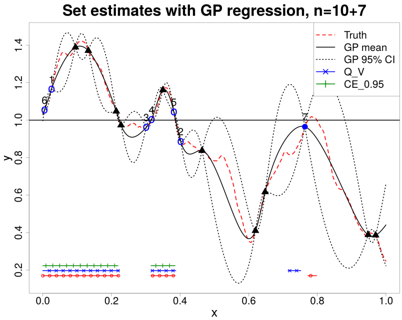

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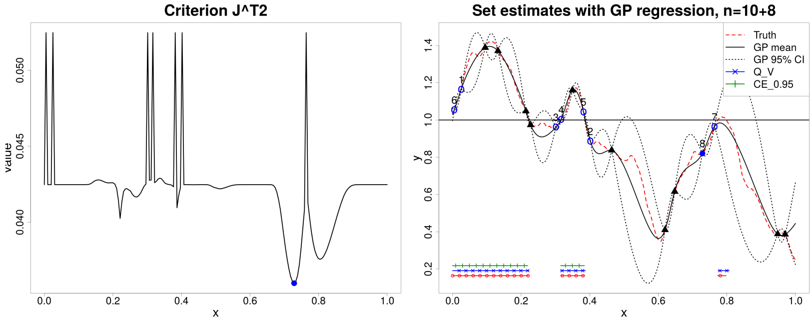

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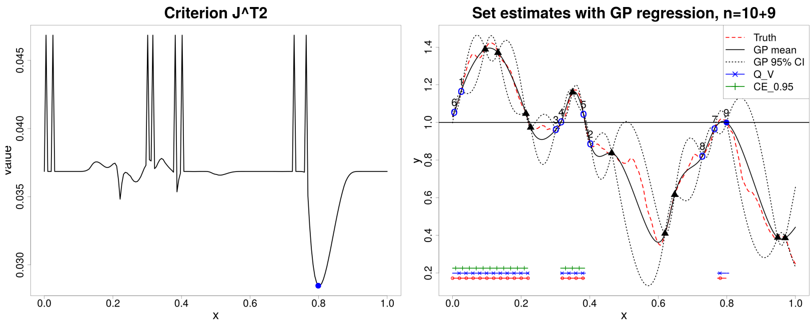

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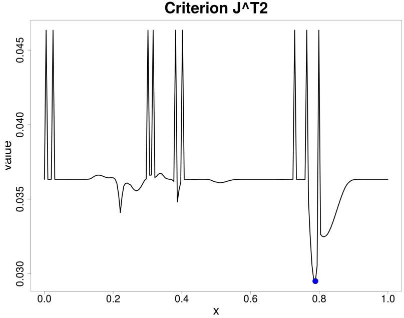
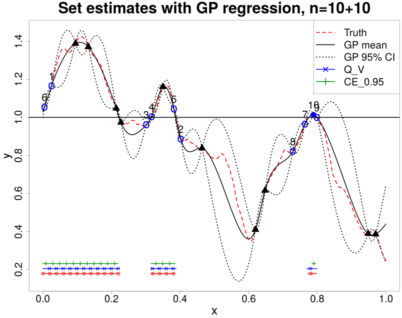
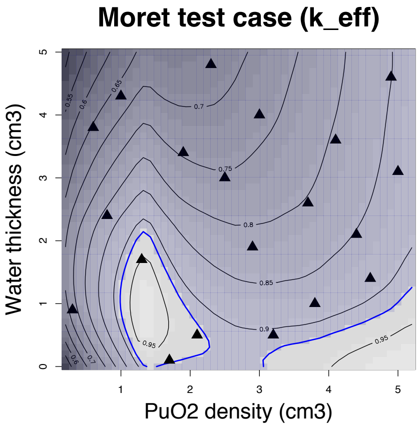




More

IRSN test case



Test case:

- $\mathrm{k_{eff}}$ function of $\mathrm{PuO_2}$ density and $\mathrm{H_2O}$ thickness, $D = [0.2, 5.2] \times [0, 5]$;
- $\mathrm{k_{eff}}$ continuous, expensive to evaluate;
- $n = 20$ observations (black triangles);

Objective: estimate $\Gamma^\star = \{x \in D : f(x) \leq t\}$ and evaluate the uncertainty of the estimate.

GRF model: m constant, k Matérn ($\nu = 5/2$), MLE for hyper-parameters.

Acknowledgements: Yann Richet, Institut de Radioprotection et de Sûreté Nucleaire.

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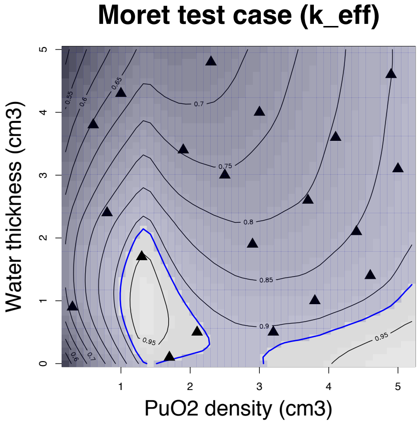

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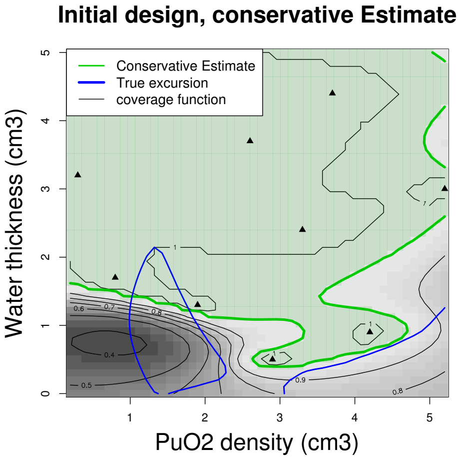
IRSN test case: type II uncertainty



Criterion: $J_{n+q}^{\mathrm{T2}}(\mathbf{x}_q) = \mathbb{E}_n[\mu(\Gamma \setminus Q_{n+1,\rho_n^{\alpha}}) \mid \mathbf{X}_{n+q} = \mathbf{x}_q]$,

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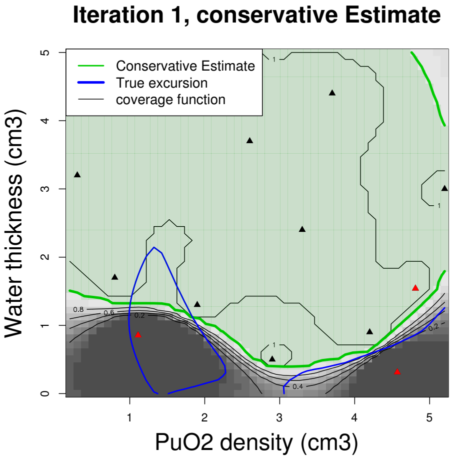

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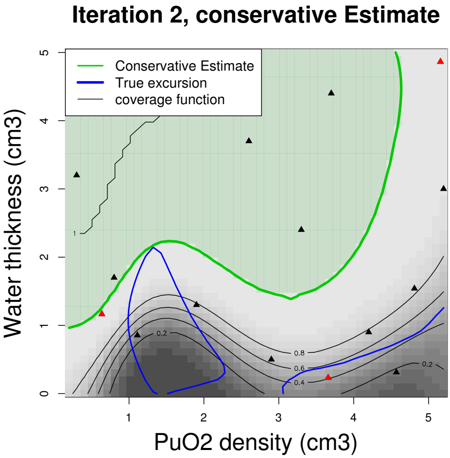

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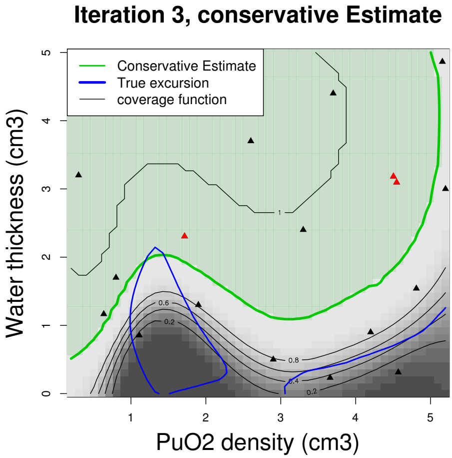

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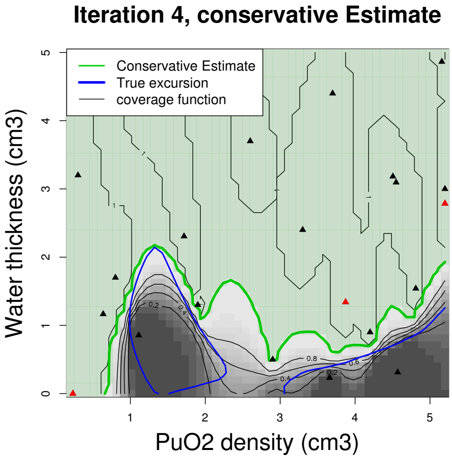

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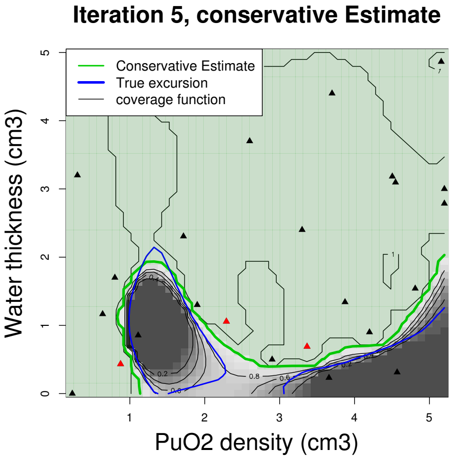

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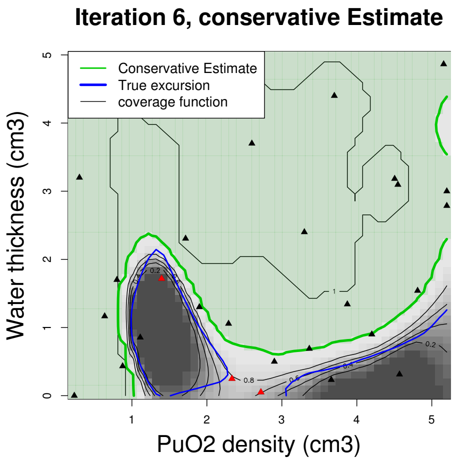

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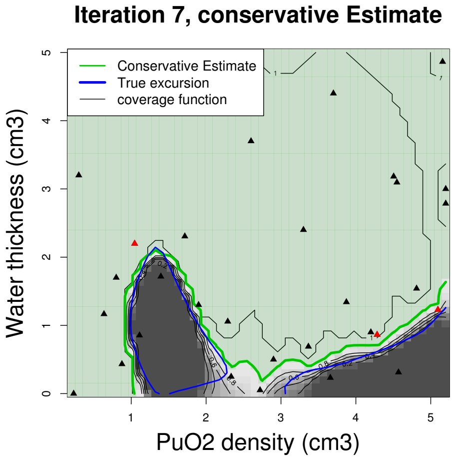

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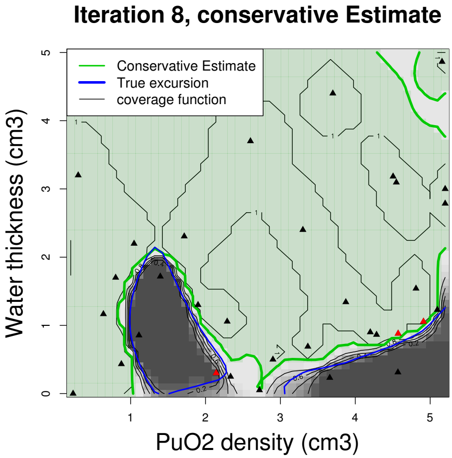

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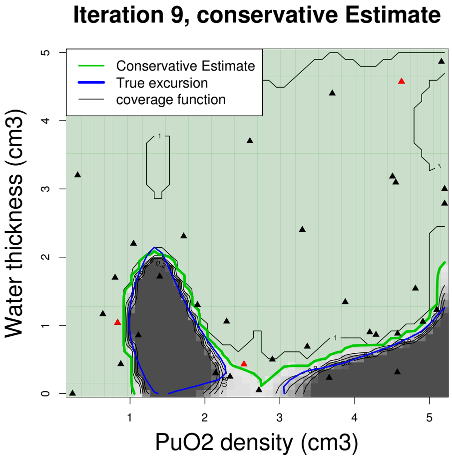

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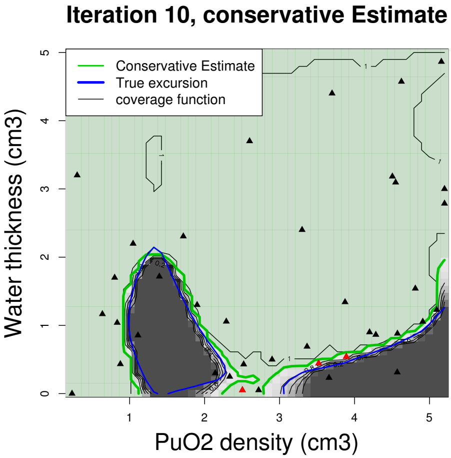

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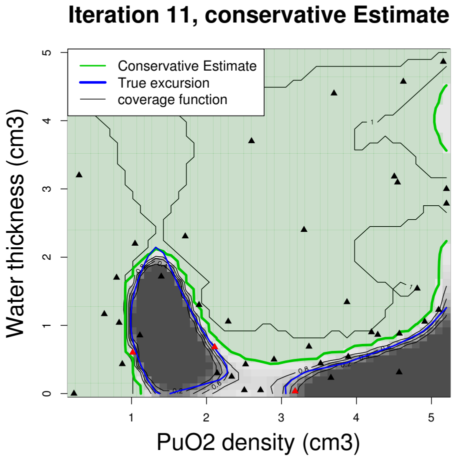

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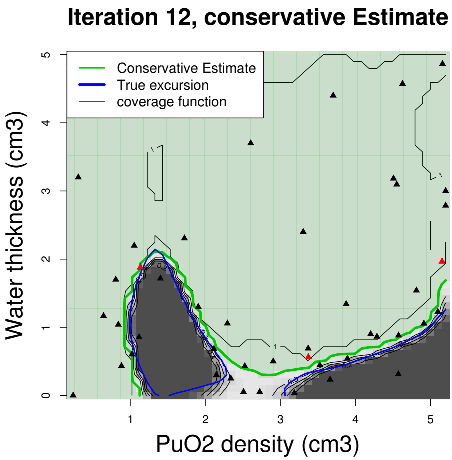

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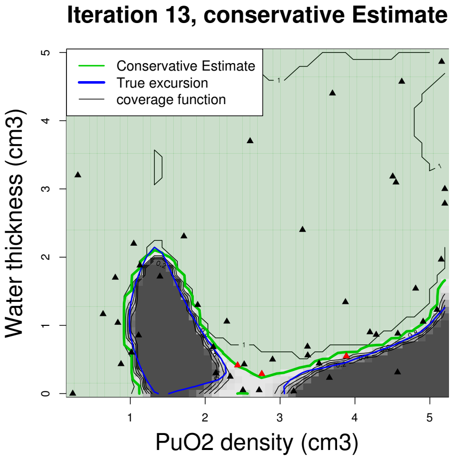

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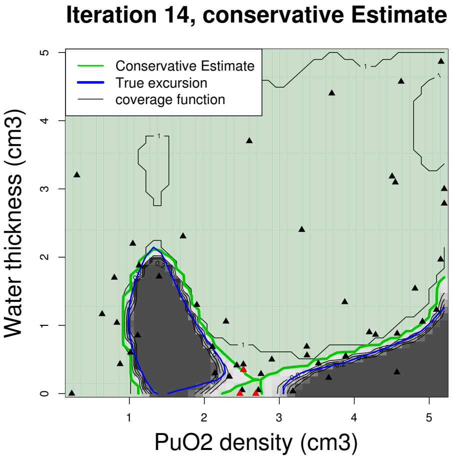

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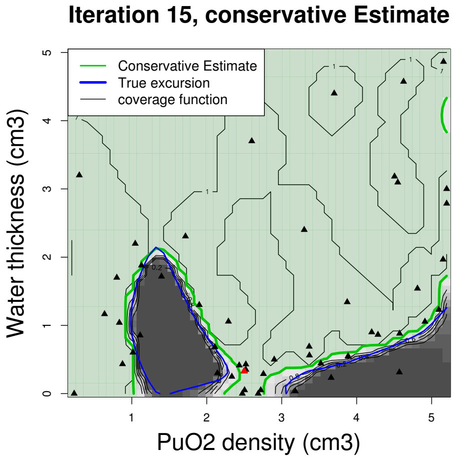

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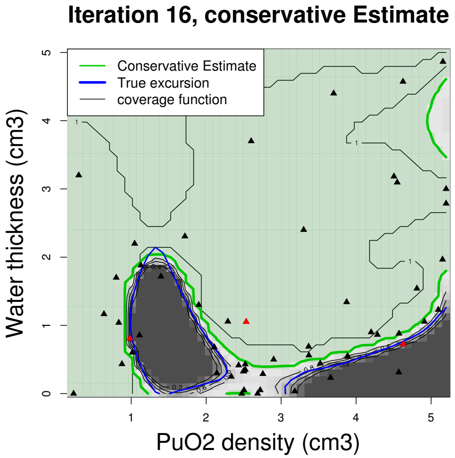

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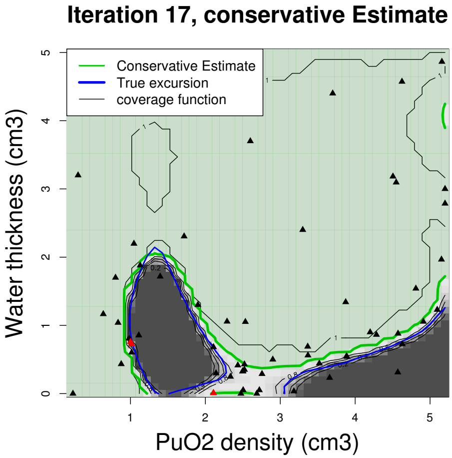

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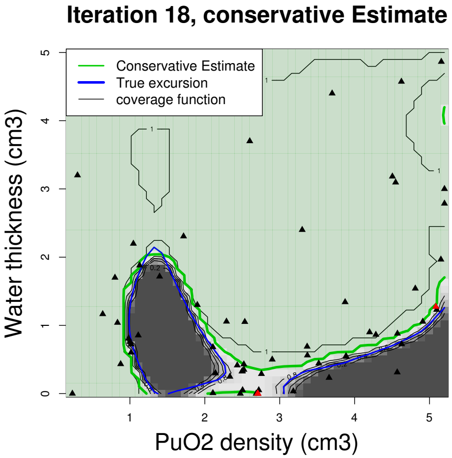

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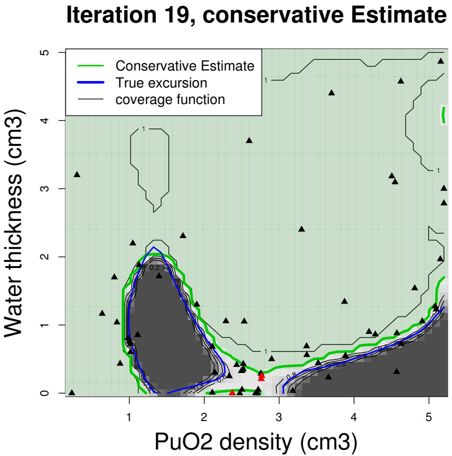

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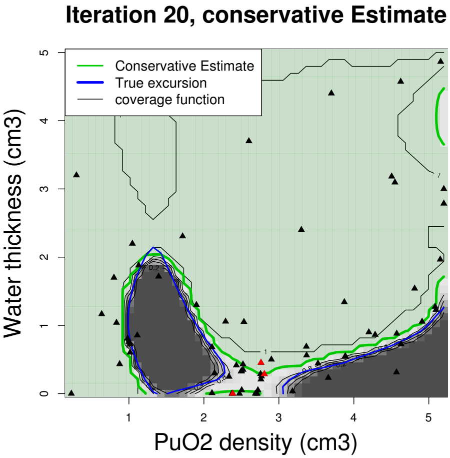

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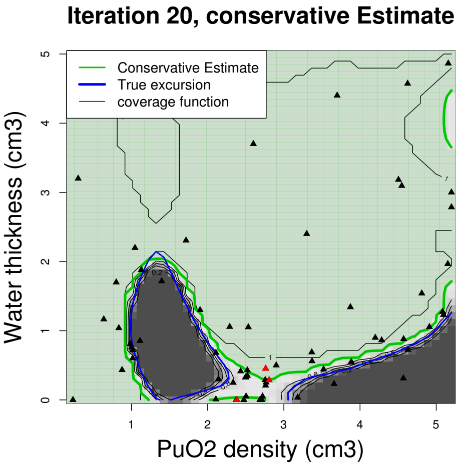


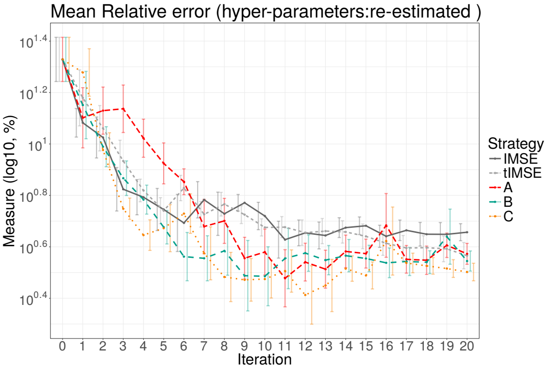
$n = 60$ new evaluations;

Next evaluation chosen in order to minimize criterion $J_n^{\mathrm{T}2}$;

Volume of updated CE: 21.30 (true excursion: 22.04, initial estimate: 18.10)

Comparison with state-of-the-art

IRSN test case: m constant, k Matérn ($\nu = 5/2$).



$n_0 = 10$ initial points;

$q = 3$ points per iteration;

Relative error with respect to 50×50 evaluation grid.

Strategy	Criterion
IMSE	
tIMSE	target=0.92
A	$J_n(\cdot; \rho_n)$, $\rho_n = 0.5$
B	$J_n(\cdot; \rho_n^*)$, $\alpha = 0.95$
C	$J_n^{\mathrm{T2}}(\cdot; \rho_n^*)$, $\alpha = 0.95$

A., D., Ginsbourger, D., Chevalier, C., Bect, J. and Richet, Y. (2017+). *Adaptive design of experiments for conservative estimation of excursion sets.* Under revision. hal-01379642.

Outline

Visualization techniques

Problem: how to visualize sets in dimensions higher than 3?

For high dimensional data

- dimensionality reduction methods
  (PCA, multidimensional scaling, ...)
- projection pursuit, Tours
- parallel coordinates
- pair plots

Those methods are well suited for sets of points, however we lose the concept of set and often the boundaries are deformed.

Visualization techniques

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  (PCA, multidimensional scaling, ...)
- projection pursuit, Tours
- parallel coordinates
- pair plots

Those methods are well suited for sets of points, however we lose the concept of set and often the boundaries are deformed.

Visualization of excursion sets

Focus on excursion sets of the form

$$\Gamma = \{x \in D : f(x) \geq t\}$$

where $D \subset \mathbb{R}^d$, with $d \geq 3$.

Coordinate profile extrema functions

Consider, for $i \in \{1, \ldots, d\}$, $\mathbf{e}_i$, the i-th canonical coordinate vector. The *i-th coordinate profile extrema* functions are

$$P_i^{\sup}\gamma : \ \eta \in E_{\mathbf{e}_i} \longrightarrow \ P_i^{\sup}\gamma(\eta) := \sup_{x_i=\eta} \gamma(\mathbf{x})$$

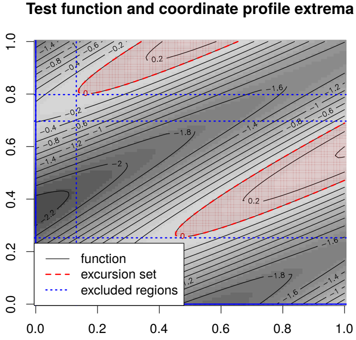
$$P_i^{\inf}\gamma : \ \eta \in E_{\mathbf{e}_i} \longrightarrow \ P_i^{\inf}\gamma(\eta) := \inf_{x_i=\eta} \gamma(\mathbf{x}).$$

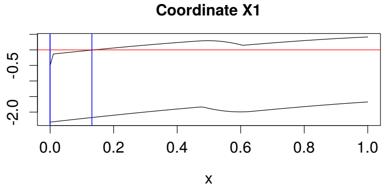
where $E_{\mathbf{e}_i} = \{\eta \in \mathbb{R} : \mathbf{e}_i^T \mathbf{x} = \eta, \ \mathbf{x} \in D\}$ and $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_d)$.

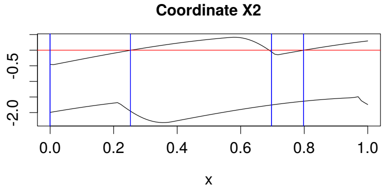
Analytical test case

$$\gamma(\mathbf{x}) = \sin\left(a{\mathbf{v_1}}^T\mathbf{x} + b\right) + \cos\left(c{\mathbf{v_2}}^T\mathbf{x} + d\right) \qquad \mathbf{x} \in [0,1]^2,\ \mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^2$$

where $a, b, c, d \in \mathbb{R}$, $\mathbf{v_1} = [\cos(\theta), \sin(\theta)]^T$, $\mathbf{v_2} = [\cos(\theta + \pi/2), \sin(\theta + \pi/2)]^T$ and $\theta = \pi/6$. *Threshold* $t = 0$.







A generalization

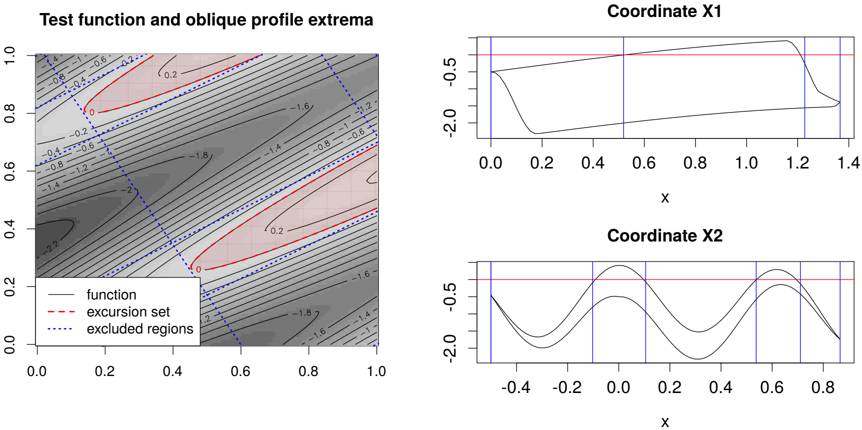
Profile extrema functions

Consider for a fixed matrix $\Psi \in \mathbb{R}^{d \times p}$ the functions

$$P_{\Psi}^{sup} f(\eta) = \sup_{x : \Psi^T x = \eta} f(x) \qquad P_{\Psi}^{inf} f(\eta) = \inf_{x : \Psi^T x = \eta} f(x),$$

- In the previous example, with $\Psi = \mathbf{v}_1 \in \mathbb{R}^2$ then $P_{\Psi}^{sup} f(\eta)$ is the sup of f under the constraint $\mathbf{v}^T x = \eta$.
- If $d > 2$ we can consider *bivariate profile extrema* where $\Psi \in \mathbb{R}^{d \times 2}$.

Oblique profile extrema

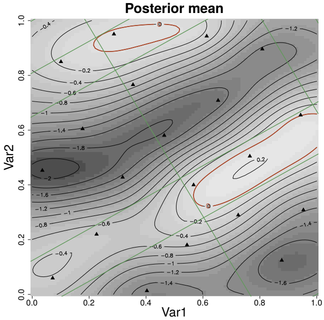


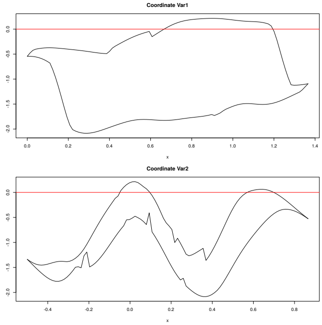
Profile extrema functions on posterior mean

Consider the analytical test case above, assume that only $n = 20$ evaluations are available.

Model: GP with prior constant mean, Matèrn covariance kernel with $\nu = 5/2$, $n = 20$ points. MLE for hyper-parameters.

Set of interest: $\Gamma = \{x \in D : m_n(x) \geq t\}$ with $t = 0$.





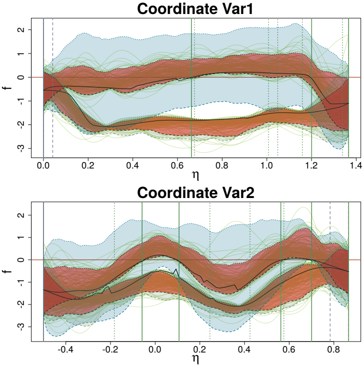
Evaluate the uncertainty on profile extrema

- Approximated posterior realizations

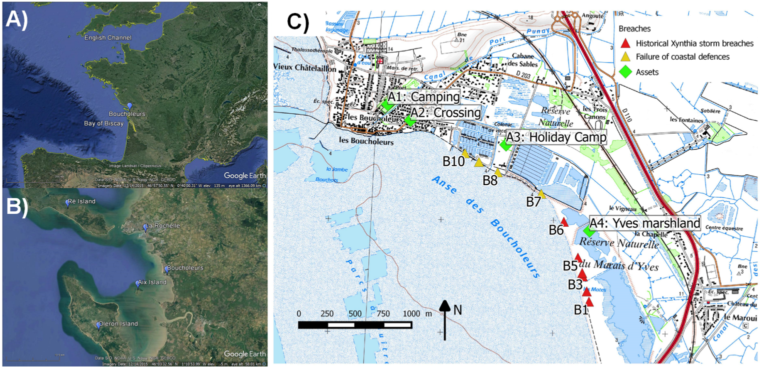
$$\widetilde{Z}_x = a(x) + \mathbf{b}^T(x) Z_E \quad (x \in D), \qquad (1)$$

where $E = \{e_1, \ldots, e_m\}$, $a : D \longrightarrow \mathbb{R}$ is a trend function and $\mathbf{b} : D \longrightarrow \mathbb{R}^m$ is a vector-valued function of deterministic weights. More;

- Compute profile extrema for each realization;
- Point-wise confidence intervals (red).
- probabilistic bound gives conservative confidence intervals (light blue) that account for approximation in eq. 1. More;



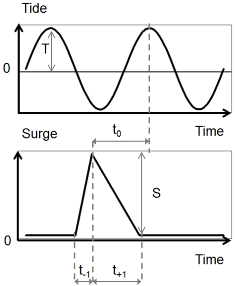
Boucholeurs test case



Numerical simulations validated on flooding caused by Xynthia storm (BRGM). More

Boucholeurs test case

$(x_i, y_i) \in D \times \mathbb{R}$, $i = 1, \dots, n$ with $y_i = f(x_i)$. $n = 200$, $D = [0,1]^5$.
$f : D \subset \mathbb{R}^5 \to \mathbb{R}$ expensive to evaluate (around 30 mins).



Inputs:

Tide (x_1): high tide level;

Surge (x_2): the storm surge magnitude peak;

Phi (x_3): phase difference tide and surge;

tMinus (x_4): time duration surge increase;

tPlus (x_5): time duration surge decrease;

Response: Number of cells flooded (25m). Square-root transformed.

GP model

Model: Gaussian process with

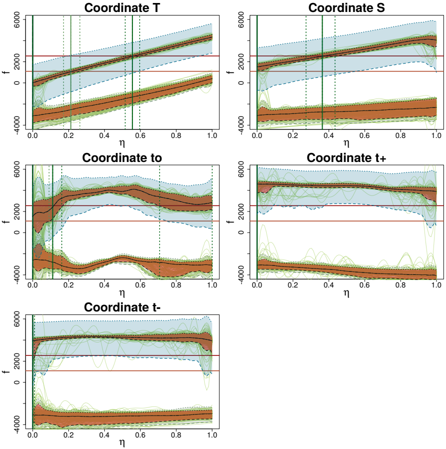
- $n = 200$;
- prior mean

$$m(\mathbf{x}) = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3^2 + c_4 x_4 + c_5 x_5.$$

- Matérn ($\nu = 3/2$) covariance kernel;
- unknown homogeneous noise estimated from data;
- MLE for hyper-parameters and trend.

Set of interest: $\Gamma = \{x \in D : m_n(x) \geq t\}$ with $t = \sqrt{1.2 \times 10^6 \text{ m}^2}$ and $t = \sqrt{6.5 \times 10^6 \text{ m}^2}$, where $D \subset \mathbb{R}^5$ and m_n posterior mean.

Profile extrema functions (5-d)

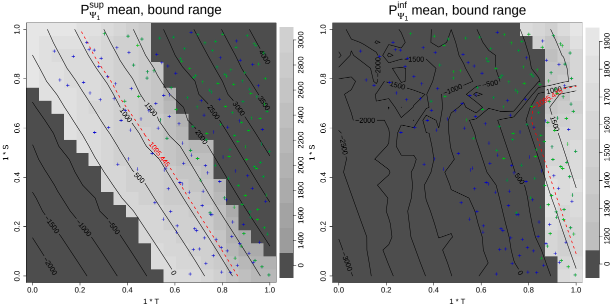


Profile extrema:

- posterior mean (black solid);
- point-wise confidence intervals (90%, red).
- conservative bound ($\alpha = 5\%$, blue)

Bivariate profile extrema

Profile extrema functions with $\Psi_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$



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Conclusion

- Conservative estimates:
  - Reduced the computing time for conservative estimates;
  - Methods implemented in R package anMC.
- Sequential strategies for conservative estimates:
  - Stepwise Uncertainty Reduction framework;
  - Comparison with state-of-the-art strategies.
- Visualization of excursion regions with profile extrema:
  - analysis tool providing lower dimensional subsets of excursion;
  - Implemented in R package profExtrema

Thanks for your attention!

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References

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Appendix

SUR strategies formulae

Criterion J_n

The criterion J_n can be expanded in closed-form as

$$
\begin{aligned}
J_n(\mathbf{x}^{(q)};\rho_n) &= \mathbb{E}_{n+q}\left[\mu\left(\Gamma\Delta Q_{n+q,\rho_n}\right) \mid X_{n+1}=\mathbf{x}^{(\mathbf{q})}_{\mathbf{1}},\dots,X_{n+q}=\mathbf{x}^{(\mathbf{q})}_{\mathbf{q}}\right] \\
&= \int_D \left(2\Phi_2\left(\begin{pmatrix} a_{n+q}(u) \\ \Phi^{-1}(\rho_n)-a_{n+q}(u)\end{pmatrix};\begin{pmatrix} 1+\gamma_{n+q}(u) & -\gamma_{n+q}(u) \\ -\gamma_{n+q}(u) & \gamma_{n+q}(u)\end{pmatrix}\right)\right. \\
&\quad \left. - p_n(u) + \Phi\left(\frac{a_{n+q}(u)-\Phi^{-1}(\rho_n)}{\sqrt{\gamma_{n+q}(u)}}\right)\right) d\mu(u), \text{where}
\end{aligned}
$$

$$
a_{n+q}(u)=\frac{m_n(u)-t}{s_{n+q}(u)},\qquad \mathbf{b}_{n+q}(u)=\frac{K_q^{-1}k_n^T(u,\mathbf{x}^{(\mathbf{q})})}{s_{n+q}(u)},\ \gamma_{n+q}(u)=\mathbf{b}^T_{n+q}(u)K_q\mathbf{b}_{n+q}(u),
$$

with $u\in D$, $k_n(u,\mathbf{x}^{(\mathbf{q})})=(k_n(u,x_{n+1}),\dots,k_n(u,x_{n+q}))$,
$K_q=[k_n(x_{n+i},x_{n+j})]_{i,j=1,\dots,q}$, $\Phi_2(\cdot;\Sigma)$ bivariate centred Normal distribution with covariance matrix Σ and $\Phi^{-1}(u)$ quantile at level u of the standard Normal distribution.

SUR strategies formulae

Criterion J_n^{T2}

The criterion $J_n^{\mathrm{T2}}(\cdot;\rho_n^\alpha)$ can be expanded in closed-form as

$$J_n^{\mathrm{T2}}(\mathbf{x}^{(\mathbf{q})};\rho_n^\alpha) = \mathbb{E}_{n+q}\left[G_n^{(2)}(Q_{n+q,\rho_n^\alpha}) \mid X_{n+1} = \mathbf{x}_{\mathbf{1}}^{(\mathbf{q})}, \ldots, X_{n+q} = \mathbf{x}_{\mathbf{q}}^{(\mathbf{q})}\right]$$
$$= \int_D \Phi_2\left(\begin{pmatrix} a_{n+q}(u) \\ \Phi^{-1}(\rho_n^\alpha) - a_{n+q}(u)\end{pmatrix}; \begin{pmatrix} 1+\gamma_{n+q}(u) & -\gamma_{n+q}(u) \\ -\gamma_{n+q}(u) & \gamma_{n+q}(u)\end{pmatrix}\right) d\mu(u)$$

where

$$a_{n+q}(u) = \frac{m_n(u) - t}{s_{n+q}(u)}, \qquad \mathbf{b}_{n+q}(u) = \frac{K_q^{-1} k_n^T(u, \mathbf{x}^{(\mathbf{q})})}{s_{n+q}(u)}, \ \gamma_{n+q}(u) = \mathbf{b}_{n+q}^T(u) K_q \mathbf{b}_{n+q}(u),$$

with $u \in D$, $k_n(u, \mathbf{x}^{(\mathbf{q})}) = (k_n(u, x_{n+1}), \ldots, k_n(u, x_{n+q}))$,
$K_q = [k_n(x_{n+i}, x_{n+j})]_{i,j=1,\ldots,q}$, $\Phi_2(\cdot;\Sigma)$ bivariate centred Normal distribution with covariance matrix Σ and $\Phi^{-1}(u)$ quantile at level u of the standard Normal distribution.

Back