Gaussian Random Field: simulation and quantification of the error

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EMSE

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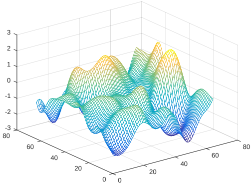

1. Continuity
   - Separability
   - Proving continuity without separability
2. Simulation
   - Exact simulation
   - The stationary case
   - Algorithm and quantification of the error
3. Application
   - Medical framework
   - Simulation
   - Example of non-stationary gaussian random field

Definition

Definition

A real valued random field ϕ on $(\Omega, \mathcal{A}, \mathbf{P})$ indexed by a metric space (M, d) is called separable, if there exists an at most countable subset S of M which is dense in (M, d), so that for all closed intervals C in $\mathbb{R}$, and all open subsets O of M,

$$\{\phi(x) \in C, x \in O\} = \{\phi(x) \in C, x \in O \cap S\}.$$

holds. Then S is called a separating set for ϕ.

Alternative definition

Lemma

A real valued random field ϕ on $(\Omega, \mathcal{A}, \mathbf{P})$ indexed by (M, d) is separable with separating set S if and only if one of the following equivalent statements holds :

(S_1) *For every open subset O in M,*

$$\inf_{y \in O \cap S} \phi(y) = \inf_{x \in O} \phi(x),$$
$$\text{and} \quad \sup_{y \in O \cap S} \phi(y) = \sup_{x \in O} \phi(x) \; ;$$

(S_1') *For every $x \in M$,*

$$\liminf_{y \to x,\ y \in S} \phi(y) = \liminf_{y \to x} \phi(x),$$
$$\text{and} \quad \liminf_{y \to x,\ y \in S} \phi(y) = \liminf_{y \to x} \phi(x).$$

Definitions

- Let M be a compact of $\mathbb{R}^d$. (Or $M = [-N, N]^d$, $N \in \mathbb{N}^*$) ;
- $$D_n \overset{\text{def}}{=} \left\{ \frac{k}{2^n},\ k \in \mathbb{Z}^d \right\} \cap M,$$
- We define M the following norm : for all $x, y \in M$,
$$d(x, y) \overset{\text{def}}{=} \sup_{i \in 1,\dots,d} |x_i - y_i|.$$
- We will denote $\delta_n \overset{\text{def}}{=} \frac{1}{2^n}$.

Properties

We have

$$D_n \subset D_{n+1}$$

and

$$|D_n| \leq \left(2^n \mathrm{diam}(M) + 1\right)^d,$$

For $x \in D_n$ we define ($\delta_n (= \frac{1}{2^n})$)

$$C_n(x) \stackrel{\text{def}}{=} \{y \in D_n,\ d(x,y) \leq \delta_n\}.$$

We have the following upperbound

$$C_n(x) \leq 3^d.$$

We define also

$$\pi_n \stackrel{\text{def}}{=} \{\{x, y\},\ x, y \in D_n,\ d(x, y) \leq \delta_n\}.$$

Then

$$|\pi_n| \leq |D_n| \sup_{x \in D_n} |C_n(x)| \leq \left(2^n \mathrm{diam}(M) + 1\right)^d 3^d.$$

($\mathcal{D}$) For all $n > 1$ and x, $y \in D_n$, there exists x', $y' \in D_{n+1}$ such that

- $d(x, x') \leq \delta_{n+1}$ et $d(y, y') \leq \delta_{n+1}$;
- $d(x', y') \leq d(x, y)$.

Conditions for the random field

We finally set

$$D = \bigcup_{n \in \mathbb{N}} D_n,$$

$(\mathcal{C})$ there exist two nondecreasing functions q and r such that

$$\sum_{n=1}^{\infty} |\pi_n| q(\delta_n) < +\infty \qquad (\mathcal{C}_1)$$

$$\sum_{n=1}^{\infty} r(\delta_n) < +\infty \qquad (\mathcal{C}_2)$$

$$\mathbf{P}\left(|\phi(x) - \phi(y)| \geq r\big(d(x,y)\big)\right) \leq q\big(d(x,y)\big), \qquad (\mathcal{C}_3)$$

for all x, $y \in M$ with $d(x, y) < \rho$.

Choice of r

We want r being "as big as possible", along the condition $(\mathcal{C}_2)$: we choose (with $\rho < 1$ and $\alpha > 1$)

$$r(h) \overset{\text{def}}{=} \left(\frac{1}{\ln_2(1/h)}\right)^{\alpha} = \left(\frac{1}{|\ln_2(h)|}\right)^{\alpha}, \qquad 0 < h < 1 \qquad (1)$$

since then

$$r(\delta_n) = r\left(\frac{1}{2^n}\right) = \left(\frac{1}{\ln_2(2^n)}\right)^{\alpha} = \frac{1}{n^{\alpha}},$$

Choice of q

$$q(h) = h^{(1+\beta)d}$$

With

$$|\pi_n| \leq \left(2^n \mathrm{diam}(M) + 1\right)^d 3^d \\ \leq (6\mathrm{diam}(M))^d 2^{nd}$$

we have $q(\delta_n) = \frac{1}{2^{(1+\beta)dn}}$ and

$$|\pi_n| q(\delta_n) \leq (6\mathrm{diam}(M))^d \frac{2^{nd}}{2^{(1+\beta)dn}} = C_d \frac{1}{(2^{\beta d})^n}$$

since $\beta d > 0$, $2^{\beta d} > 1$, leads to a convergent series.

Fundamental Lemma

Lemma

Under the conditions $(\mathcal{D})$ *and* $(\mathcal{C})$, *there exists a set* $\Omega' \subset \Omega$, $\mathbf{P}(\Omega') = 1$, *such that for all* $\omega \in \Omega'$ *there exists* $n(\omega) \in \mathbb{N}$ *such that*

1. *for all* $n \geq n(\omega)$,
$$\max_{(x,y)\in\pi_n} |\phi(x,\omega) - \phi(y,\omega)| \leq r(\delta_n); \quad (2)$$
2. *for all* $m \geq n \geq n(\omega)$, *and every* $x, y \in \mathcal{D}_m$ *with* $d(x,y) \leq \delta_n$ *we have*
$$|\phi(x,\omega) - \phi(y,\omega)| \leq 2\sum_{k=n}^{m} r(\delta_k). \quad (3)$$

We can state the continuity with respect to the set $\mathcal{D}$

Proposition

Under the conditions $(\mathcal{D})$ *and* $(\mathcal{C})$ *there exists a set* $\Omega' \subset \Omega$, $\mathbf{P}(\Omega') = 1$, *such that for all* $\omega \in \Omega'$ *the restriction of the function* $x \mapsto \phi(x, \omega)$ *to* $\mathcal{D}$ *is (uniformly) continuous.*

Theorem

Let ϕ be continuous in probability so that the conditions $(\mathcal{D})$ and $(\mathcal{C})$ hold. Then, there exists $\Omega' \subset \Omega$, $\mathbf{P}(\Omega') = 1$, such that we have a uniformly sample continuous modification $\tilde{\phi}$ of ϕ such that $\tilde{\phi} = \phi$ on $D \cap \Omega'$.

Lemma

Suppose that the random field ϕ satisfies condition $(\mathcal{C}_3)$ with

$$\lim_{x \to 0} q(x) = \lim_{x \to 0} r(x) = 0$$

(which is true, if the three conditions $(\mathcal{C})$ are satisfied). Then ϕ is continuous in probability.

Exact simulation

- Let S_n be a finite index set (size $n \times n$) associated with a random field $\longrightarrow$ Gaussian vector of size n^2, denoted by X.
- The covariance C is then a $n^2 \times n^2$-matrix.
- We know that if Z is a Gaussian vector of size n^2, which the componants have i.i.d. normal distributions (a « white noise ») and A is such that $A^2 = C$, then X has the same law as AZ.

Examples with Gaussian covariance

The matlab function "simgauss(n)" computes a square of size $(2^n + 1) \times (2^n + 1)$.
We have the following duration times :

» simgauss(2) ;
Elapsed time is 0.066492 seconds.
» simgauss(3) ;
Elapsed time is 0.244321 seconds.
» simgauss(4) ;
Elapsed time is 2.839962 seconds.
» simgauss(5) ;
Elapsed time is 44.630163 seconds.

Examples with Gaussian covariance

» *simgauss(6) ; Elapsed time is 771.891478 seconds.*

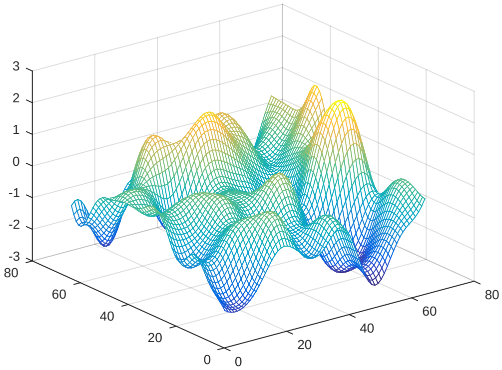

FIGURE: Gaussian kernel, variance : 0.01

Examples with Gaussian covariance

» simgauss(7) ; Elapsed time is 8551.463134 seconds.

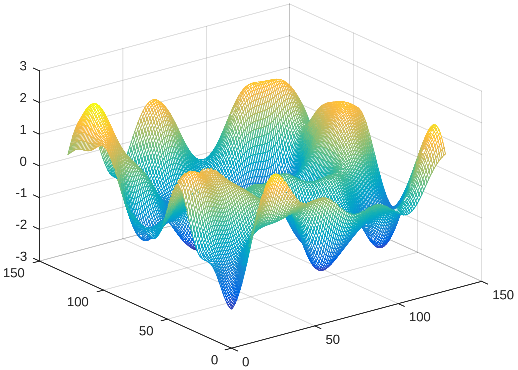

FIGURE: Gaussian kernel, variance : 0.01

Examples with Gaussian covariance

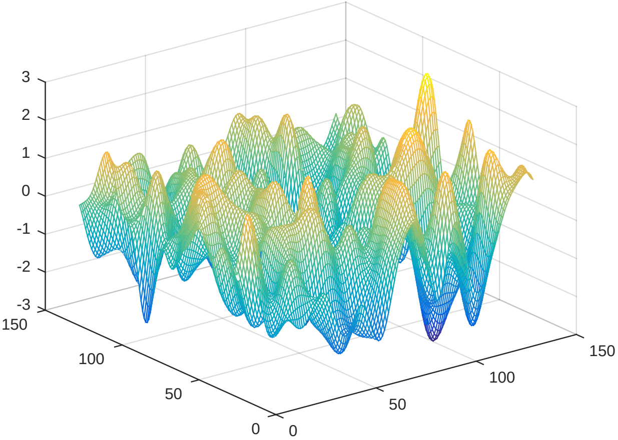

FIGURE: Gaussian kernel, variance : 0.05

» *simgauss(7) ; Elapsed time is 8475.642908 seconds.*

Examples with Gaussian covariance

The time is not the only issue of the naïve algorithm :

» simgauss(8) ; Requested 66049x66049 (32.5GB) array exceeds maximum array size preference. Creation of arrays greater than this limit may take a long time and cause MATLAB to become unresponsive. See array size limit or preference panel for more information.

Generalized Random Field

We consider the complexification of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\mathcal{S}_{\mathbb{C}}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and denote by $L^0_K(P)$ the space of real-or complex valued random variables, with $K = \mathbb{R}$ or $\mathbb{C}$.

Definition

A generalized random field on $\mathbb{R}^d$ is a K-linear mapping

$$\varphi : \mathcal{S}_K(\mathbb{R}^d) \mapsto L^0_K(P)$$

where $K = \mathbb{R}$ or $\mathbb{C}$.

Theorem (Minlos' theorem)

Let Ξ be a characteristic function on $\mathcal{S}(\mathbb{R}^d)$, i.e.

1. *Ξ is continuous in $\mathcal{S}(\mathbb{R}^d)$,*
2. *Ξ is positive definite,*
3. *$\Xi(0) = 1$.*

Then there exists a unique probability measure μ on $(\mathcal{S}'(\mathbb{R}^d), \sigma_w)$, such that for all $f \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i\langle \omega, f\rangle} d\mu(\omega) = \Xi(f)$$

i.e. $\Xi(f)$ is the Fourier transform of a countably additive positive normalized measure.

Generalized white noise

Definition

A generalized random field W on $\mathbb{R}^d$ is called white noise, if its characteristic function is given by

$$\Xi_{\mathrm{WN}}(f) = e^{-\frac{1}{2}\langle f,f\rangle_{L_2(R^d)}}.$$

Remark : A real-valued white noise on $\mathbb{R}^d$, is a linear mapping $W : \mathcal{S}(\mathbb{R}^d) \mapsto L^0(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is some probability space, so that the family $\{W(f),\ f \in \mathcal{S}(\mathbb{R}^d)\}$ is a centered Gaussian family with

$$Cov(W(f), W(g)) = E[W(f)W(g)] = \langle f, g\rangle_{L_2(R^d)}.$$

Proposition

W extends to an isometric embedding of $L^2(\mathbb{R}^d)$ into $L^2(P)$.

Let φ be a generalized random field. We define operations via dual pairing :

Definition

1. *The Fourier transform F and its inverse F^{-1} are defined on φ via*
$$\mathcal{F}\varphi(f) := \varphi(\mathcal{F}^{-1}f)$$
$$\mathcal{F}^{-1}\varphi(f) := \varphi(\mathcal{F}f)$$
2. *If $g \in \mathcal{C}^\infty(\mathbb{R}^d)$ with at most polynomial growth, then*
$$g\varphi(f) := \varphi(\overline{g}f)$$

Construction of Stationary Gaussian Random Fields

Let φ be a stationary, real-valued, centered Gaussian random field with covariance

$$Cov(\varphi(f), \varphi(g)) = \langle f, Cg \rangle_{L_2(R^d)}, \quad f, g \in \mathcal{S}(\mathbb{R}^d)$$

with $C : \mathcal{S}(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d)$. Assume that C is given by an integral kernel, which by stationarity can be written as

$$Cf(x) = \int_{\mathbb{R}^d} K(x-y) f(y) dy,$$

where K is even because of the symmetry of C, and positive definite.

Hence if K is continuous,

$$K(x) = \int e^{-2\pi ixp} d\Gamma(p) \qquad \text{(Böchner Theorem)}$$

- We assume that Γ has a density γ which is supposed to be strictly positive and smooth. Then $\gamma^{\frac{1}{2}}$ is a strictly positive smooth root of γ.
- We set

$$\varphi(f) := (\mathcal{F}^{-1}\gamma^{\frac{1}{2}}\mathcal{F}W)(f), \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

Discrete Fourier transform

- $N_1, \ldots, N_d$ will represent even positive integers.
- For all $u = (u_1, \ldots, u_d) \in \prod_{i=1}^{d} [\![0, N_i - 1]\!]$:

$$\mathcal{F}_{N_1,\ldots,N_d}(f)(u) := \frac{1}{N_1 \cdots N_d} \sum_{\substack{0 \le k_1 \le N_1 - 1 \\ \vdots \\ 0 \le k_d \le N_d - 1}} e^{2i\pi\left(\frac{u_1 k_1}{N_1} + \cdots + \frac{u_d k_d}{N_d}\right)} f(k_1, \ldots, k_d). \quad (4)$$

We will denote by $\mathcal{F}^S_{(N_i)}$ the "symmetric" discrete Fourier transform

$$\mathcal{F}^S_{N_1,\ldots,N_d}(g)(v) := \frac{1}{N_1 \cdots N_d} \sum_{\substack{-\frac{N_1}{2} \le k_1 \le \frac{N_1}{2} - 1 \\ \vdots \\ -\frac{N_d}{2} \le k_d \le \frac{N_d}{2} - 1}} e^{2i\pi\left(\frac{v_1 k_1}{N_1} + \cdots + \frac{v_d k_d}{N_d}\right)} g(k_1, \ldots, k_d).$$

$$\mathrm{sym}_{(N_i)} : \left\{ \begin{array}{lcl} \prod_{i=1}^{d} [\![0, N_i]\!] & \to & \prod_{i=1}^{d} [\![-\frac{N_i}{2}, \frac{N_i}{2}]\!] \\ (x_i) & \mapsto & (y_i) \text{ with } \left[\begin{array}{lll} y_i = x_i & \text{if} & x_i \in [\![0, \frac{N_i}{2} - 1]\!] \\ y_i = x_i - N_i & \text{if} & x_i \in [\![\frac{N_i}{2}, N]\!] \end{array} \right. \end{array} \right. \tag{6}$$

and, conversely,

$$\mathrm{sym}^{-1}_{(N_i)} : \left\{ \begin{array}{lcl} \prod_{i=1}^{d} [\![-\frac{N_i}{2}, \frac{N_i}{2}]\!] & \to & \prod_{i=1}^{d} [\![0, N_i]\!] \\ (y_i) & \mapsto & (x_i) \text{ with } \left[\begin{array}{lll} x_i = y_i & \text{if} & y_i \in [\![0, \frac{N_i}{2} - 1]\!] \\ x_i = y_i + N_i & \text{if} & y_i \in [\![-\frac{N_i}{2}, 0]\!] \end{array} \right. \end{array} \right. \tag{7}$$

So, while working on a function $c : \prod_{i=1}^{d} [\![-\frac{N_i}{2}, \frac{N_i}{2} - 1]\!] \to \mathbb{R}$, we will use the function

$$\tilde{c}(x) := c\big(\mathrm{sym}_{(N_i)}(x)\big), \qquad \forall x \in \prod_{i=1}^{d} [\![0, N_i]\!].$$

Lemma

$$\mathcal{F}^S_{(N_i)}(f)(x) = \mathcal{F}_{(N_i)}(f \circ \mathrm{sym}_{(N_i)})(\mathrm{sym}^{-1}_{(N_i)}(x)).$$

We can state the lemma on which will be built the algorithm :

Lemma

Let $E =: \prod_{i=1}^{d} [\![0, N_i]\!]$, $h : E \to \mathbb{C}$ and $(A_p)_{p\in E}$, $(B_q)_{q\in E}$ be two independent white noises. The centered Gaussian Random field indexed on $\prod_{i=1}^{d} [\![0, \frac{N_i}{2}]\!]$ defined by

$$G(x) := \Re\left(\mathcal{F}^{-1}_{(N_i)}((A + iB)h)(x)\right) \quad (8)$$

has the following covariance function :

$$\mathrm{E}\left[G(x)G(y)\right] = \Re\left(\mathcal{F}^{-1}_{(N_i)}(|h|^2)(\mathrm{sym}^{-1}_{(N_i)}(y - x))\right) \quad (9)$$

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Heuristic of the Algorithm

- Let $c : (\mathbb{R}^n)^2 \to \mathbb{R}$ be a (continuous) covariance function (*i.e.* symmetric definite positive) and we assume that function to be stationary (*i.e.* invariant with respect to translation).
- Notation : $c(h) := c(0, h)$, $h \in \mathbb{R}^n$.

1. This last function, according to a consequence of the Böchner's Theorem has a real positive valued Fourier's transform.
2. On the other hand, if c has a compact support, or if its tails norm decrease sufficiently fast to 0, the fast Fourier transform can be seen as an approximation of its Fourier's transform.

Using those two facts, we can consider that the values of the discrete Fourier transform of c are real and (close to being) positive and, with $h(x) := \sqrt{\mathcal{F}_{(N_i)}(c)(x)} \in \mathbb{R}^+$, the Lemma 5 provides a Gaussian Random Field with covariance c

$$\begin{aligned} \mathrm{E}\left[G(x)G(y)\right] &= \Re\left(\mathcal{F}^{-1}_{(N_i)}\left(\left|h\right|^2\right)(y-x)\right) \\ &= \Re\left(\mathcal{F}^{-1}_{(N_i)}\left(\mathcal{F}_{(N_i)}(c)\right)(y-x)\right) \\ &= \Re\left(c(x-y)\right) \\ &= c(x-y) = c(x,y). \end{aligned}$$

Remark

In the general case (i.e. $\mathcal{F}_{(N_i)}(c)(x) \in \mathbb{R}$, because of the symmetry of c) we have

$$\mathrm{E}\left[G(x)G(y)\right] = \Re\left(\mathcal{F}^{-1}_{(N_i)}\left(\left|\mathcal{F}_{(N_i)}(c)\right|\right)(y-x)\right).$$

Unidimensional case

The idea is to approximate N values in $[-\frac{1}{2}, \frac{1}{2}]$ of a Fourier transform of a function f (assuming that the function of interest has negligible values outside of $[-\frac{1}{2}, \frac{1}{2}]$) by

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\frac{x}{N}y} f(y)\,\mathrm{d}y, \qquad x \in \left[\!\left[-\frac{N}{2}, \frac{N}{2} - 1 \right]\!\right]. \tag{10}$$

then by

$$\frac{1}{N}\mathcal{F}_N^s(f_N)(x) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} e^{ix\frac{k}{N}} f_N(k) \tag{11}$$

(with $f_N(k) \stackrel{\text{def}}{=} f(\frac{k}{N})$).

Error quantification

Proposition

Let $N \in 2\mathbb{N}$, $c : E \to \mathbb{C}$ and $(A_p)_{p \in E}$, $(B_q)_{q \in E}$ be two independent white noises. The centered Gaussian Random field indexed on $[\![0, \frac{N}{2}]\!]$ defined by

$$G(x) := \Re\left(\mathcal{F}^{-1}_{(N_i)}\left((A + iB)\sqrt{\mathcal{F}_N(c)(x)}\right)(x)\right) \tag{12}$$

has a covariance function c_G such that, for all $x, y \in [\![0, \frac{N}{2}]\!]$,

$$|c(x, y) - c_G(x, y)| \leq 2n_N(\varepsilon_c + \varepsilon_N) \tag{13}$$

where n_N stands for the numbers of x such that $\mathcal{F}_N(c)(x) < 0$ and with ε_c such that

$$\sup_{x \in \left[-\frac{1}{2}, \frac{1}{2}\right]} \left| \int_{\mathbb{R} \setminus \left[-\frac{1}{2}, \frac{1}{2}\right]} e^{i2\pi xy} c(y)\, \mathrm{d}y \right| \leq \varepsilon_c.$$

with

$$\varepsilon_N \leq \frac{|c(\frac{1}{2})|}{N} + \sum_{k=1}^{p} \frac{b_{2k}}{N^{2k}(2k)!}(2\pi)^{2k-1}2^{2k+1}\|c\|_{2k+1,i}\left(\tfrac{1}{2}\right) + \frac{1}{N^{2p+1}(2p+1)}R_{N,p} \quad (14)$$

where $\|c\|_{2k+1,o}(x) \stackrel{\text{def}}{=} \sum_{\substack{l=1 \\ l \text{ impair}}}^{2k+1} \left|c^{(l)}(x)\right|$ and with

$$\left|R_{n,p}\right| \leq (4\pi)^{2p+1} \sup_{\substack{t\in[-\frac{1}{2},\frac{1}{2}] \\ l\in[\![0,2p+1]\!]}} \left|c^{(l)}(t)\right| \int_0^1 \left|B_{2p+1}(u)\right| du, \quad (15)$$

where B_i (resp. $b_i \stackrel{\text{def}}{=} B_i(0)$) are the Bernoulli polynomials (resp. Bernoulli numbers).

Example : the exponential covariance case

$$c(x) \stackrel{\text{def}}{=} e^{-\frac{x^2}{2\sigma^2}}.$$

we have (with $p = 0$)

$$|c(x,y) - c_G(x,y)| \leq 2n_N \underbrace{\left(e^{-\frac{1}{8\sigma^2}} \left(4 + \frac{1}{N}\right) + \frac{1}{2N}\left(\frac{1}{\sigma^2} + 2\pi\right)\right)}_{e_N}.$$

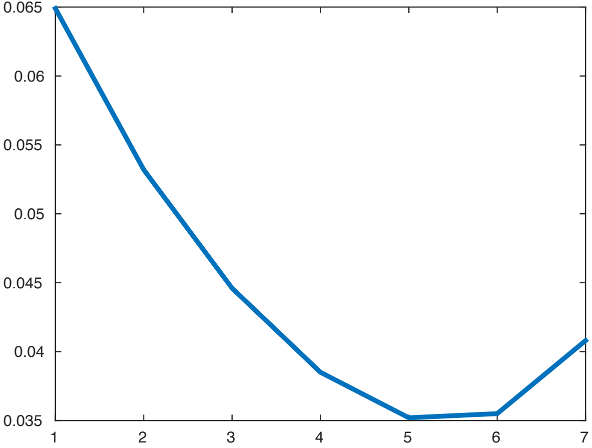
- For $N = 1000$ we have :

σ	0.09	0.1	0.11	0.12	0.13	0.14	0.15
e_{1000}	0.0649	0.0532	0.0446	0.0385	0.0352	0.0355	0.0408

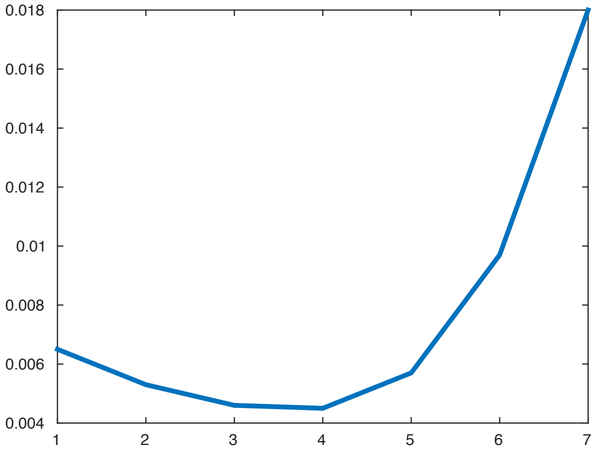
- For $N = 10000$

σ	0.09	0.1	0.11	0.12	0.13	0.14	0.15
e_{10000}	0.0065	0.0053	0.0046	0.0045	0.0057	0.0097	0.0180

Example, the exponential covariance case : error (p=0, N=1000)



Example, the exponential covariance case : error (p=0, N=10000)



Example, the exponential covariance case

For $p = 1$, it follows

$$\varepsilon_N \leq \frac{\left|c(\frac{1}{2})\right|}{N} + \frac{b_2}{2N^2} 2\pi 2^3 \left(\left|c'(\tfrac{1}{2})\right| + \left|c'''(\tfrac{1}{2})\right| \right) + \frac{1}{3N^3} R_{N_3}$$
$$\leq e^{-\frac{1}{8\sigma^2}} \left(\frac{1}{N} + \frac{2\pi}{3N^2} \left(\frac{1}{\sigma^2} + \frac{3}{\sigma^4} + \frac{1}{4\sigma^6} \right) \right) + \frac{1}{N^3} \left(\frac{1}{\sigma^2} + \frac{3}{2\sigma^4} + \frac{1}{8\sigma^6} \right)$$

- For $N = 1000$ we have :

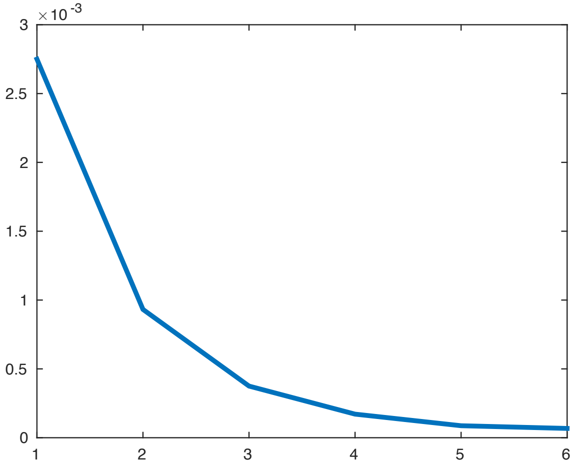
σ	0.04	0.05	0.06	0.07	0.08	0.09
e_{1000}	0.0104	0.00275	9.32×10^{-4}	3.751×10^{-4}	1.713×10^{-4}	8.71×10^{-5}

and for $\sigma = 0.095$, $e_{1000} = 6.7484 \times 10^{-5}$.

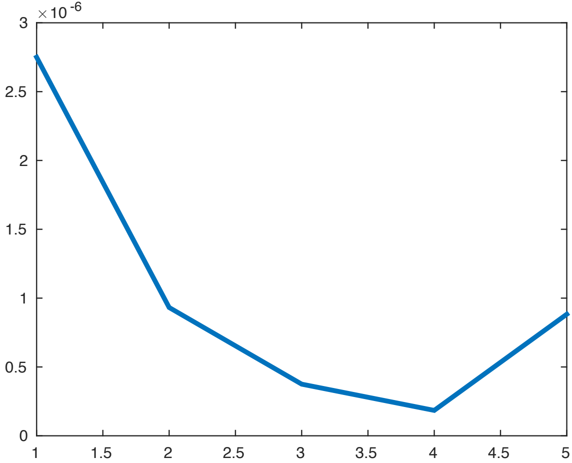
- For $N = 10000$

σ	0.05	0.06	0.07	0.08	0.09
e_{10000}	2.75×10^{-6}	9.32×10^{-7}	3.751×10^{-7}	1.845×10^{-7}	8.83×10^{-7}

Example, the exponential covariance case : error (p=1, N=1000)



Example, the exponential covariance case : error (p=1, N=10000)



The compact support case

When the covariance function has a compact support included in $[-\frac{1}{2}, \frac{1}{2}]$, the speed can be improved (especially when the derivatives are uniformly bounded), since we have for some $p \in \mathbb{N}^*$, if c is a $\mathcal{C}^p$ function,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\frac{x}{N}y} f(y)\,\mathrm{d}y - \frac{1}{N}\mathcal{F}_N^s(f_N)(x) = \frac{R_{N,p}}{N^{2p+1}(2p+1)}$$

with $\|R_{n,p}\| \leq M_{2p+1} \int_0^1 |B_{2p+1}(t)|\,\mathrm{d}t$.

The multidimensional case

We consider a continuous covariance function $c : \mathbb{R}^2 \to \mathbb{R}$ belonging to $\mathrm{L}^1(\mathbb{R}^2)$.
Let ε_c be such that

$$\int_{\mathbb{R}^2 \setminus [-\frac{1}{2}, \frac{1}{2}]^2} |c(u, v)| \, du \, dv \leq \varepsilon_c.$$

$$
\begin{aligned}
&\left| \int_{[-\frac{1}{2},\frac{1}{2}]^2} g(u,v)\,\mathrm{d}u\,\mathrm{d}v - \frac{1}{MN}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} g(u_k,v_l)\right| \\
\leq & \frac{1}{2N}\sup_{v\in[-\frac{1}{2},\frac{1}{2}]}\left|g\left(\tfrac{1}{2},v\right)-g\left(-\tfrac{1}{2},v\right)\right| + \frac{1}{2M}\sup_{u\in[-\frac{1}{2},\frac{1}{2}]}\left|g\left(u,\tfrac{1}{2}\right)-g\left(u,-\tfrac{1}{2}\right)\right| \\
&+\sum_{k=1}^{p}\frac{b_{2k}}{(2k)!}\Bigg(\frac{1}{N^{2k}}\sup_{v\in[-\frac{1}{2},\frac{1}{2}]}\left|g^{(2k-1)}\left(\tfrac{1}{2},v\right)-g^{(2k-1)}\left(-\tfrac{1}{2},v\right)\right| \\
&\qquad\qquad + \frac{1}{M^{2k}}\sup_{u\in[-\frac{1}{2},\frac{1}{2}]}\left|g^{(2k-1)}\left(u,\tfrac{1}{2}\right)-g^{(2k-1)}\left(u,-\tfrac{1}{2}\right)\right|\Bigg) \\
&+\frac{1}{N^{2p+1}(2p+1)}\sup_{v\in[-\frac{1}{2},\frac{1}{2}]} R^v_{N,p} + \frac{1}{M^{2p+1}(2p+1)}\sup_{u\in[-\frac{1}{2},\frac{1}{2}]} R^u_{M,p} \\
=:& \varepsilon_{N,M}.
\end{aligned}
$$

Example : the exponential covariance case

$$c(u,v) = e^{-\frac{u^2+v^2}{2\sigma^2}}$$

$$g(u,v) = c(u,v)e^{2i\pi\left(u\frac{x}{N}+v\frac{y}{M}\right)}$$

$$\begin{aligned} g_1'(u,v) &= -\frac{u}{\sigma^2}e^{-\frac{u^2+v^2}{2\sigma^2}}e^{2i\pi\left(u\frac{x}{N}+v\frac{y}{M}\right)} + 2i\pi\frac{x}{N}g(u,v) \\ &= g(u,v)\Big(-\frac{u}{\sigma^2}+2i\pi\frac{x}{N}\Big). \end{aligned}$$

$$\begin{aligned} g_1''(u,v) &= g(u,v)\Big(-\frac{u}{\sigma^2}+2i\pi\frac{x}{N}\Big)^2 + \frac{-1}{\sigma^2} \\ &= g(u,v)\bigg(\Big(-\frac{u}{\sigma^2}+2i\pi\frac{x}{N}\Big)^2 - \frac{1}{\sigma^2}\bigg) \end{aligned}$$

And,

$$g_1'''(u,v) = g(u,v)\Big(\Big(-\frac{u}{\sigma^2}+2i\pi\frac{x}{N}\Big)^3 - \frac{3}{\sigma^2}\Big(-\frac{u}{\sigma^2}+2i\pi\frac{x}{N}\Big)\Big)$$

It leads to the following upperbound :

$$|g_1'''(u,v)| \leq |c(u,v)|\Big(4\frac{|u|^3}{\sigma^6}+4(2\pi)^3+3\frac{|u|}{\sigma^4}+6\pi\Big)$$

Exemples:

- For $N = 1024$ we have the following results :

σ	0.1	0.11	0.12	0.13	0.14	0.15
e^2_{1024}	0.00097	0.00055	0.00033	0.000212	0.000183	0.000336
σ	0.16	0.17	0.18	0.19	0.2	
e^2_{1024}	0.000995	0.0029	0.0073	0.019	0.031	

The human eye

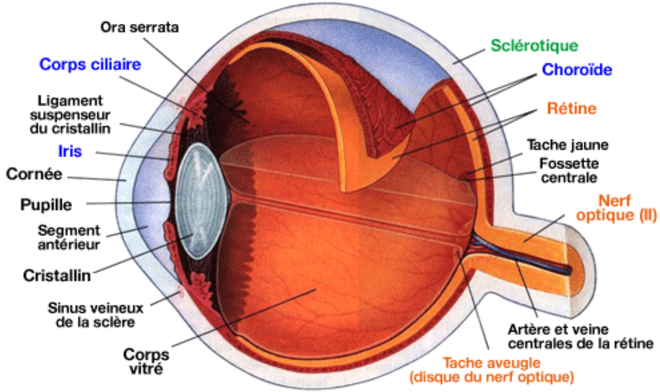
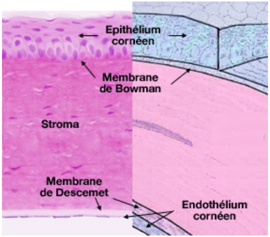


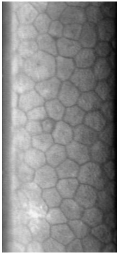
FIGURE: L'œil en coupe

Corneal endothelium

Definition (Corneal endothelium)

It separates the cornea from the aqueous humor. It is a cellular monolayer comprising about 500,000 cells whose quality and quantity vary with age.





Morphology

The endothelium is composed of relatively regular hexagonal (or almost) cells.

However, there are some more pronounced irregularities :

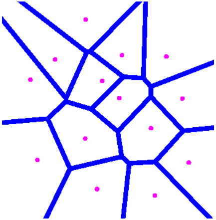
- on the edges (larger cells) ;
- in case of pathologies (e.g. Cornea guttata) ;
- in case of transplant, . . .

Voronoï Partition

Question

How will we simulate the cells ?

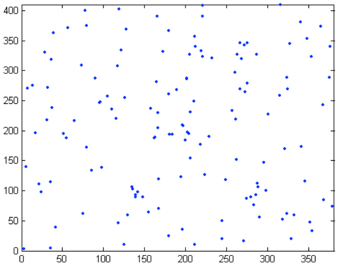
- If we already have a set of points, we can make a Voronoï partition

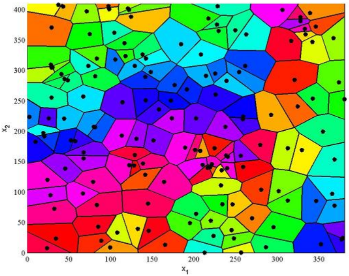


- For each point p of the surface S, the Voronoï cell $V(p)$ of p is the set of the points closer to p than any other points of S.
- The Voronoï partition $V(S)$ is the partition made by the Voronoï cells.

Poisson point process and Voronoï partion

Random space points are most commonly generated using a Poisson point process. . .





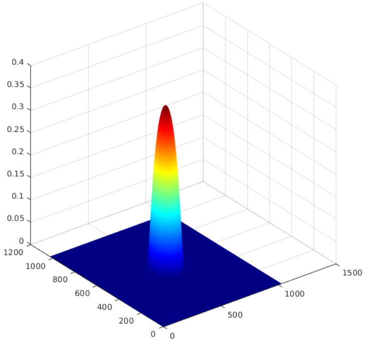
. . .*which is rather irregular*. . .

Superposition an functions with compact support

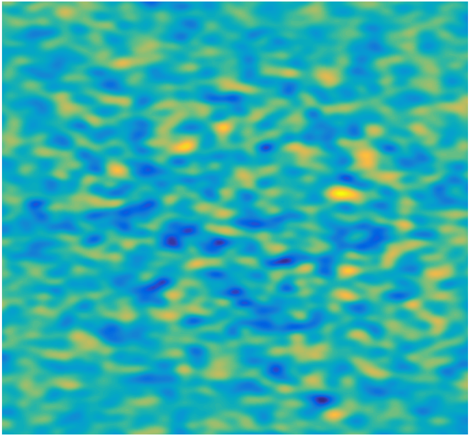
We use the combination

$$G_1(x) + f(x)G_2(x)$$

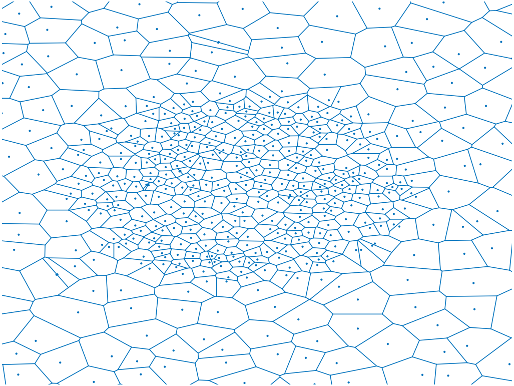
where f is a compact support function :



Non-stationary gaussian random field : "large center"



Voronoï partition with high centred density

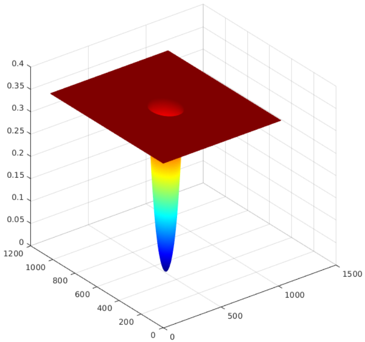


Weak local density

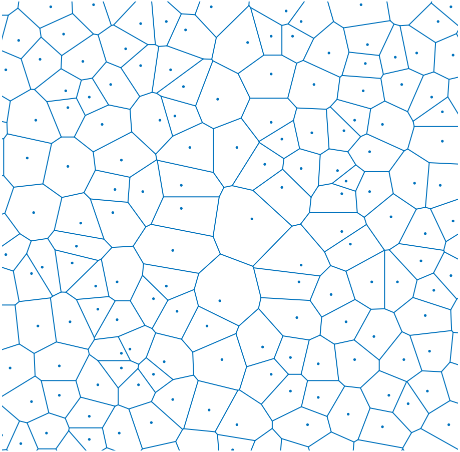
We use a combination

$$G_1(x) + f(x)G_2(x)$$

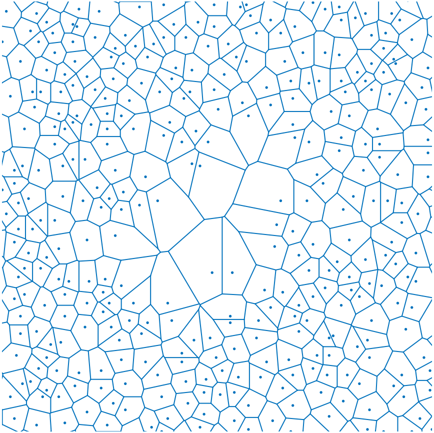
where f is the complementary of a compact support function



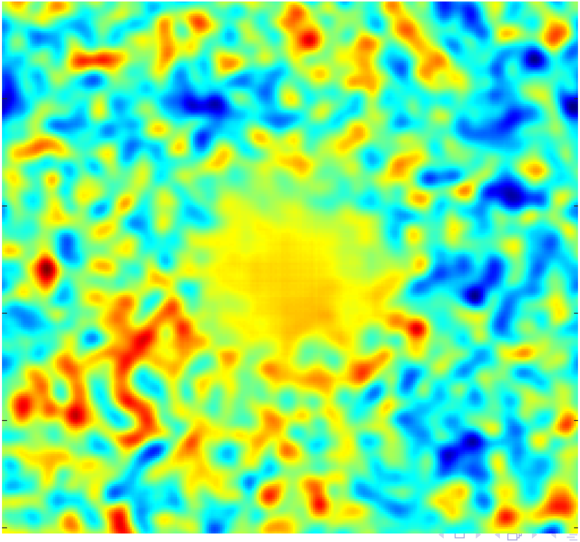
Voronoï weak local density I



Voronoï weak local density II



Random field for simulating the Cornea Guttata



Conclusion and perspective

- Quantification of the error
- and combination of Gaussian stationary random fields (with related "well chosen" functions belonging to the algebra generated by smooth functions with compact support)

provide an easy to use and interesting modelisation tool, at least in the medical field in which we had to work.

- We are working now with a combination of stationary random fields with multiplicative functions taken in wavelet basis.

Thank you for the invitation !