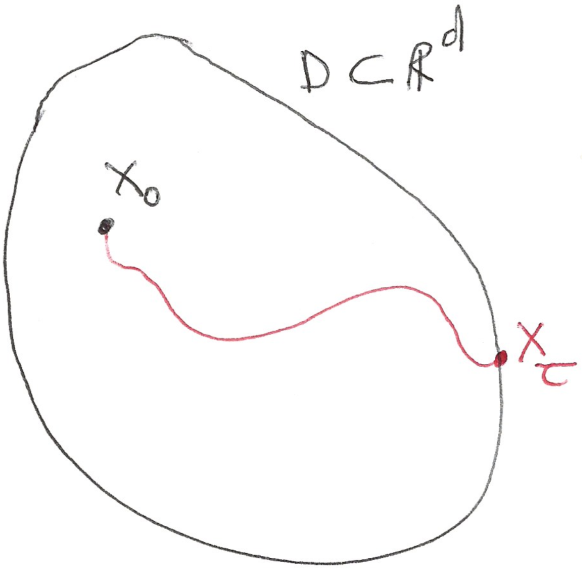
Exit-problem for self-stabilizing Diffusions

(I) Exit-problem

$X = (X_t)_{t \geq 0}$ stochastic process:
- Diffusion
- Jump process
- PDMP
- ......



$\tau := \inf\{t \geq 0 : X_t \notin D\}$.

Questions:

* Exit-time: τ? Law of τ? $\mathbb{E}[\tau]$?

* Exit-location: X_τ? Law of X_τ? Does it belong to ∂D?

Examples: τ depends on a small parameter σ.

$\Longrightarrow$ • $\frac{\sigma^2}{2} \log(\tau) \xrightarrow[\sigma \to 0]{\mathbb{P}} H > 0$.

or • $\frac{\sigma^2}{2} \log(\mathbb{E}(\tau)) \xrightarrow[\sigma \to 0]{} H > 0$; or $\mathbb{E}(\tau) \simeq k(\sigma)\, e^{\frac{2}{\sigma^2} H}$.

or $X_\tau \xrightarrow[\sigma \to 0]{\mathbb{P}} x^* \in \partial D$.

(II) Exit-problem for linear diffusions

$X^\sigma = (X_t^\sigma)_{t \geq 0}$ s.t.

$$X_t^\sigma = x_0 + \sigma B_t - \int_0^t \nabla V(X_s^\sigma)\, ds$$

$x_0 \in \mathbb{R}^d$; $\sigma > 0$; $(B_t)_{t \geq 0}$ B.M.

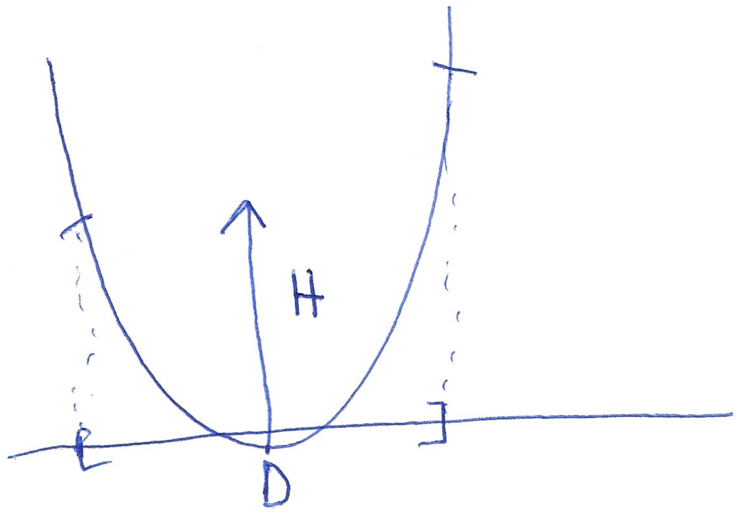
V is uniformly strictly convex : $\nabla^2 V \geq \theta > 0$.

$V \geq V(0) = 0$.

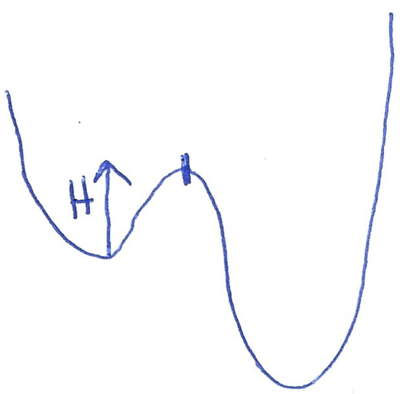
Then Freidlin-Wentzell theory (Large deviations for processes) implies:

$$\forall \delta > 0 : \quad \mathbb{P}\left(e^{\frac{2}{\sigma^2}(H-\delta)} \leq \tau_D \leq e^{\frac{2}{\sigma^2}(H+\delta)} \right) \xrightarrow[\sigma \to 0]{} 1$$

with $H := \inf_{\partial D} V$.



If V is not convex:



<u>References</u>: Book of Dembo/Zeitouni

Potential theory (see the papers of Bovier)

implies $\mathbb{E}[\tau_D] = \frac{K}{\sigma} e^{\frac{2H}{\sigma^2}}(1+o(1))$

III Self-Stabilizing Diffusions

1) Introduction

Kac (1959): simplification of kinetic Vlasov equation on plasmas:

$$X_t^{i,N} = X_0^i + \sigma B_t^i - \int_0^t \nabla V(X_s^{i,N})\,ds - \frac{1}{N}\sum_{j=1}^N \int_0^t \nabla F(X_s^{i,N} - X_s^{j,N})\,ds$$

* $1 \leq i \leq N$.
* $(X_0^i)_i$ iid.
* $\sigma > 0$.
* $(B^i)_i$: B.M. ⫫ and ⫫ from $(X_0^i)_i$.
* V: confining potential.
* F: interacting potential.

When $N \longrightarrow \infty$:

$$\begin{cases} X_t^{1,\infty} = X_0^1 + \sigma B_t^1 - \int_0^t \nabla V(X_s^{1,\infty})\,ds - \int_0^t \nabla F * \mu_s^{1,\infty}(X_s^{1,\infty})\,ds \\ \mu_t^{1,\infty} = \mathcal{L}(X_t^{1,\infty}) \end{cases}$$

This limit is called "propagation of chaos".

More precisely: $\lim_{N\infty} \sup_{t\in[0;T]} \mathbb{E}\left[\|X_t^{1,\infty} - X_t^{1,N}\|^2\right] = 0;\ \forall T>0.$

Example: $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) = \frac{\alpha}{2}x^2$, $\alpha > 0$. Then:

$$X_t = X_0 + \sigma B_t - \int_0^t \left(X_s^3 + (\alpha-1)X_s - \alpha\,\mathbb{E}(X_s)\right) ds$$

Interpretation: a lot of economical agents which interact.

2) Some results (V, F convexes)

* **Mckeon (66/67)** : Existence of a unique strong solution.

Moreover : $\frac{\partial}{\partial t}\mu_t = \operatorname{div}\left\{\frac{\sigma^2}{2}\nabla\mu_t + \mu_t \nabla W_t\right\}$; $W_t := V + F * \mu_t$.

* ~~BRT~~ Benachour, Roynette, Talay and Vallois (98)
  Benachour, Roynette and Vallois (98)

Existence, invariant probability, convergence, propagation of chaos.

* Benedetto, Caglioti, Carrillo, Pulvirenti (98) : convergence with $\nabla^2 V \geq \theta > 0$
* Carrillo, McCann, Villani (03, 06) : convergence with convexity assumptions.
* Cattiaux, Guillin, Malrieu (08) : Uniform propagation of chaos + convergence, with convexity assumptions.
* Herrmann, Imkeller, Peithmann (08) : Large deviations + exit problem with $\nabla^2 V \geq \theta > 0$.

3) Some results (V non convex)

* Herrmann, Tugaut (10) : Non uniqueness of invariant probability measure
* Tugaut (13) : Convergence to one of the invariant probability measures
* Duong, Tugaut (15, 18) : Extension to kinetic equation.
* Tugaut (21) : Stability around an invariant probability measure.
* Durmus, Guillin, Monmarché (21, preprint) : rate of convergence if σ not small.

IV Exit-time for Self-stabilizing Diffusions.

1) If V is convex

(a) HIP (AAP)

$$\frac{\sigma^2}{2}\log(\tau) \xrightarrow[\sigma\to 0]{\mathbb{P}} \hat{H} := \inf_{\partial D}\left(V + F * \delta_0\right), \quad F \text{ convex.}$$

Idea: Taking advantage of the uniqueness of the invariant probability measure + reconstructing F-W theory.

(b) Tug 12 (EJP)

Same result.

Idea:

* $\tau_{X^{1,\infty},D} \simeq \tau_{X^{1,N},D}$ for $N \gg 1$.
* $\tau_{X^{1,N},D} = \tau_{(X^{1,N},\dots,X^{N,N}),D\times(\mathbb{R}^d)^{N-1}}$.
* We apply F-W theory.

(c) Tug 16 (ECP)

Same result (again).

Idea:

* $\exists T \ \forall \sigma>0 \ \forall t\geq T: \mathcal{L}(X_t)\simeq \delta_0$
* $\mathbb{P}(\tau\leq T)\xrightarrow[\sigma\to 0]{} 0$
* Coupling between $(X_t^\sigma)_{t\geq T}$ and $(Y_t^\sigma)_{t\geq T}$ with

$$Y_t^\sigma = Y_T^\sigma + \sigma(B_t - B_T) - \int_T^t \nabla V(Y_s^\sigma)\,ds - \int_T^t \nabla F(Y_s^\sigma)\,ds.$$

(d) PECDR, Duong, Monmarché, Tomasevic, Tugaut (21, preprint)

Same result with V convex and F non-convex ($\nabla V + \nabla F$ convex).
Idea is similar to Tug16.

2) If V is not convex

(a) Tug 18 (JOTP)

Kramers' law albeit with restrictions on the domain.

Idea : $$\frac{d}{dt}\mathbb{E}\left[\|X_t^\sigma - a\|^2\right] \leq -\lambda\,\mathbb{E}\left[\|X_t^\sigma - a\|^2\right] + C\sigma^2 + C'\sqrt{\mathbb{P}(\tau(\sigma)\leq t)}\,.$$

(b) Tug 19 (Alea)

No restriction on the domain but dimension 1.

(c) Tug 20 (ESAIM)

Kramers' law for the system of particles.

(d) Tug 21 (Kinetic and Related Models)

The techniques developped previously allowed us to prove that

if $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$, $F(x) = \frac{\alpha}{2}x^2$ $\left(\alpha > \frac{1}{4}\right)$, $x_0 < 0$

(for example) $\Longrightarrow$ $\mu_t^\sigma \xrightarrow[t\infty]{} \mu_\infty^\sigma$ with $\mathbb{E}(\mu_\infty^\sigma) < 0$.

(e) Jabir, Tugaut 21 (preprint, almost)

Kramers' law for the first collision-time between two SSD.
+ two particles of Mean-field system.